QUADRATIC ALGEBRAS WITH EXT ALGEBRAS GENERATED IN TWO DEGREES

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Abstract. Green and Marcos [3] call a graded $\mathbb{k}$-algebra δ-Koszul if the corresponding Yoneda algebra $Ext(\mathbb{k},\mathbb{k})$ is finitely generated and $Ext^{i,j}(\mathbb{k},\mathbb{k})$ is zero unless $j=\delta(i)$ for some function $\delta:\mathbb{N}\to\mathbb{N}$. For any integer $m\geq 3$ we exhibit a non-commutative quadratic δ-Koszul algebra for which the Yoneda algebra is generated in degrees $(1,1)$ and $(m,m+1)$. These examples answer a question of Green and Marcos. These algebras are not Koszul but are m-Koszul (in the sense of Backelin).

1. Introduction

A connected graded algebra A over a field $\mathbb{k}$ with generators in degree one is called Koszul [5] if its associated bigraded Yoneda (or Ext) algebra $E(A)=\bigoplus_{i\leq j} Ext_A^{i,j}(\mathbb{k},\mathbb{k})$ is generated as an algebra by $Ext_A^{1,1}(\mathbb{k},\mathbb{k})$. Koszul algebras are always quadratic, i.e. the elements in a minimal collection of defining relations will always be of degree two, however not every quadratic algebra is Koszul. A quadratic algebra A will fail to be Koszul if and only if $Ext_A^{i,j}(\mathbb{k},\mathbb{k})\neq 0$ for some $i<j$.

In the Yoneda algebra of a Koszul algebra, each cohomological degree is paired with only one internal degree. Green and Marcos [3] use the following definition to study algebras which have this property.

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Definition 1.1. A graded $\Bbbk$-algebra A is δ-Koszul if it satisfies the following two conditions:

(1) $\mathrm{Ext}_A^{i,j}(\Bbbk, \Bbbk) = 0$ unless $j = \delta(i)$ for a function $\delta : \mathbb{N} \to \mathbb{N}$;
(2) $E(A)$ is finitely generated as an algebra.

A Koszul algebra is then a δ-Koszul algebra for which δ is the identity function. Green and Marcos ask if there is a bound N such that if A is a δ-Koszul algebra, then $E(A)$ is generated in degrees 0 to N.

The notion of m-Koszul described in [4] and credited to Backelin [1] serves as another measure of how close a graded $\Bbbk$-algebra comes to being Koszul. The following definition of m-Koszul should not be confused with Berger's N-Koszul [2], which refers to N-homogeneous algebras with Yoneda algebras generated in degrees one and two.

Definition 1.2. A graded algebra A is called m-Koszul if $\mathrm{Ext}_A^{ij}(\Bbbk, \Bbbk) = 0$ for all $i < j \leq m$.

While any quadratic algebra is 3-Koszul, A is m-Koszul for every $m \geq 1$ if and only if A is Koszul. It is natural to ask whether a quadratic algebra could be m-Koszul for large values of m and yet still fail to be Koszul. This question was answered by Roos [8] who showed that for any integer $m \geq 3$ there exists a commutative quadratic algebra $\tilde{S}$ which is m-Koszul but not $(m+1)$-Koszul. More precisely, $E(\tilde{S})$ is finitely generated in bi-degrees $(1, 1)$ and $(m, m+1)$, but $\tilde{S}$ is not a δ-Koszul algebra because for each $i \geq m$ there is more than one value of j for which $Ext^{ij}(\Bbbk, \Bbbk)$ is nonzero.

The purpose of this paper is to show that for any integer $m \geq 3$ there exists a δ-Koszul algebra that appears to be Koszul in all cohomological degrees except for m. Like Roos' algebras $\tilde{S}$, these algebras are m-Koszul but not $(m + 1)$-Koszul. Specifically, we show that there exists a δ-Koszul algebra C of global dimension m for which the corresponding Yoneda algebra $E(C)$ is generated as an algebra in degrees $(1, 1)$ and $(m, m + 1)$. For these algebras δ is the function

$$\delta(i) = \begin{cases} i & \text{if } i < m \\ i+1 & \text{if } i = m \\ 0 & \text{if } i > m. \end{cases}$$

Our examples answer the question posed by Green and Marcos in [3]. The algebras C illustrate that there is no bound N such that the Yoneda algebra of a δ-Koszul algebra must be generated in degrees 0 to N. Moreover, the bound does not exist even if we restrict ourselves to quadratic algebras.

A quadratic algebra A is determined by a vector space of generators $V = A_1$ and an arbitrary subspace of quadratic relations $I \subset V \otimes V$. The free algebra $\Bbbk\langle V\rangle$ carries a standard grading and A inherits a grading from this free algebra. We denote by A_n the component of A degree n. For any graded algebra $A = \oplus_k A_k$, let $A[j]$ be the same vector space with the shifted grading $A[j]_k = A_{j+k}$. Throughout we assume all graded algebras A are locally finite-dimensional with $A_i = 0$ for $i < 0$ and $A_0 = \Bbbk$.

2. The algebra C

Let m be an integer greater than 2. If $m = 3$ the algebra C has 10 generators and 8 relations. If $m = 4$ then C has 12 generators and 14 relations. For $m \geq 5$, C has $3m$ generators and $4 + 3m$ relations. For $m = 3$ the homological properties of the algebra C are not new. In the proof of Lemma 2.6 we will introduce an algebra B which shares homological properties with C for $m = 4$. Consequently, we will henceforth assume that $m \geq 5$.

The algebra C is defined as follows. The generating vector space V has the basis $\bigcup_{i=1}^{m+1} S_i$ with sets $S_1 = \{n\}$, $S_2 = \{p, q, r\}$, $S_3 = \{s, t, u\}$, $S_4 = \{v, w, x_1, y_1, z_1\}$, $S_5 = \{x_2, y_2, z_2\}$, $\cdots$, $S_{m-1} = \{x_{m-4}, y_{m-4}, z_{m-4}\}$, $S_m = \{x_{m-3}, y_{m-3}\}$, and $S_{m+1} = \{x_{m-2}\}$. For all $m \geq 5$ the space of relations I contains the generators $\{np - nq, np - nr, ps - pt, qt - qu, rs - ru, sv - sw, tw - tx_1, uv - ux_1, vx_2, wx_2, x_i x_{i+1}, sv - sy_1, tw - ty_1, ux_1 - uy_1, sz_1, tz_1, uz_1, y_{i-1}x_i + z_{i-1}y_i\}$ where $i \leq m - 3$. In addition, if $m \geq 6$ then I also contains $\{z_i z_{i+1}\}$ where $i \leq m - 5$.

Remark 2.1. We have chosen this large set of generators to clarify the exactness of the resolution below. It may be possible to construct examples with fewer generators.

Notice that any basis for I is formed from certain sums of elements of S_i right multiplied by elements of S_{i+1}. This ordering on a basis of I makes C highly noncommutative. Indeed the center of C is just the field $\Bbbk = C_0$. Moreover, this ordering tells us that the left annihilator of n is zero and more generally that the left annihilator of an element of S_i is generated by sums of elements from $\Pi_{k=j}^{i-1} S_k$ for $1 \leq j \leq i-1$. The structure of I will be exploited in the proofs of Lemmas 2.2, 2.4, 2.5 and 2.6.

Our proof relies on constructing an explicit projective resolution for $\Bbbk$ as a left C-module. Let $(P^\bullet, \lambda)$ be the sequence of projective C-modules

$$P^m \xrightarrow{\lambda_m} P^{m-1} \xrightarrow{\lambda_{m-1}} P^{m-2} \quad \cdots \quad \xrightarrow{\lambda_2} P^1 \xrightarrow{\lambda_1} C \to \Bbbk$$

where $P^m = C[-m-1]$, $P^{m-1} = (C[1-m])^7$, $P^{m-2} = (C[2-m])^{16}$, $P^2 = (C[-2])^{3m+4}$, $P^1 = (C[-1])^{3m}$, and for $3 \leq i \leq m-3$, $P^i = (C[-i])^{3m+12-3i}$.

The map from C to $\Bbbk$ is the usual augmentation. For convenience we will use λ_i to denote both the map from P^i to P^{i-1} and the matrix which gives this map via right multiplication. The map λ_1 is right multiplication by the transpose of the matrix

$$(n \;\; p \;\; q \;\; r \;\; s \;\; t \;\; u \;\; z_1 \; \ldots \; z_{m-4} \;\; v \;\; w \;\; x_1 \;\; y_1 \;\; x_2 \;\; y_2 \; \ldots \; x_{m-3} \;\; y_{m-3} \;\; x_{m-2})$$

and the map λ_m is right multiplication by the matrix

$$\begin{pmatrix} 0 & 0 & 0 & 0 & np & np & -np & 0 & \cdots & 0 \end{pmatrix}.$$

The remaining maps λ_i will be defined as right multiplication by matrices given in block form, for which we will need the following components. Let

$$\alpha = \begin{pmatrix} 0 & 0 & 0 & p & 0 & 0 & -p & p & 0 \\ 0 & 0 & 0 & 0 & q & 0 & 0 & -q & q \\ 0 & 0 & 0 & 0 & 0 & r & -r & 0 & r \end{pmatrix}, \; \alpha' = \begin{pmatrix} p & 0 & 0 & -p & p & 0 \\ 0 & q & 0 & 0 & -q & q \\ 0 & 0 & r & -r & 0 & r \end{pmatrix},$$

$$\beta = \begin{pmatrix} s & -s & 0 & 0 \\ 0 & t & -t & 0 \\ u & 0 & -u & 0 \\ s & 0 & 0 & -s \\ 0 & t & 0 & -t \\ 0 & 0 & u & -u \end{pmatrix}, \beta' = \begin{pmatrix} s & -s & 0 \\ 0 & t & -t \\ u & 0 & -u \end{pmatrix}, \; \gamma = \begin{pmatrix} v & 0 \\ w & 0 \\ x_1 & 0 \\ y_1 & z_1 \end{pmatrix},$$

$$\gamma' = \begin{pmatrix} v \\ w \\ x_1 \end{pmatrix}, \; \chi_j = \begin{pmatrix} x_j & 0 \\ y_j & z_j \end{pmatrix}, \delta = \begin{pmatrix} -p & p & 0 \\ 0 & -q & q \\ -r & 0 & r \end{pmatrix}, \; \epsilon = \begin{pmatrix} s \\ t \\ u \end{pmatrix},$$

$$\zeta_j = \begin{pmatrix} z_1 & 0 & 0 & \cdots & 0 \\ 0 & z_2 & 0 & \cdots & 0 \\ 0 & 0 & z_3 & \cdots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & z_j \end{pmatrix}, \; \eta = (0, n, n, -n), \quad \text{and}$$

$$\eta' = \begin{pmatrix} 0 & n & -n & 0 \\ 0 & n & 0 & -n \end{pmatrix}.$$

The matrix defining map λ_2 has this block form

$$\begin{pmatrix} \eta' & & & & & & & & & \\ & \delta & & & & & & & & \\ & & \epsilon & & & & & & & \\ & & & \zeta_{m-5} & & & & & & \\ & & & & \beta & & & & & \\ & & & & & \gamma & & & & \\ & & & & & & \chi_2 & & & \\ & & & & & & & \ddots & & \\ & & & & & & & & \chi_{m-4} & \\ & & & & & & & & & x_{m-3} \end{pmatrix}.$$

The matrix λ_3 has the form

$$\begin{pmatrix} 0 & \eta & & & & & & & & & \\ & & \delta & & & & & & & & \\ & & & \epsilon & & & & & & & \\ & & & & \zeta_{m-6} & & & & & & \\ & & & & & \alpha' & & & & & \\ & & & & & & \beta & & & & \\ & & & & & & & \gamma & & & \\ & & & & & & & & \chi_2 & & \\ & & & & & & & & & \ddots & \\ & & & & & & & & & & \chi_{m-5} \\ & & & & & & & & & & & x_{m-4} \end{pmatrix}.$$

For $4 \leq j \leq m-3$ the matrix λ_j has the form

$$\begin{pmatrix} \eta & & & & & & & & & & \\ & \delta & & & & & & & & & \\ & & \epsilon & & & & & & & & \\ & & & \zeta_{m-j-3} & & & & & & & \\ & & & & \alpha & & & & & & \\ & & & & & \beta & & & & & \\ & & & & & & \gamma & & & & \\ & & & & & & & \chi_2 & & & \\ & & & & & & & & \ddots & & \\ & & & & & & & & & \chi_{m-j-2} & \\ & & & & & & & & & & x_{m-j-1} \end{pmatrix}.$$

Note that χ_2 is the only χ block in λ_{m-4} and that the matrix λ_{m-3} has no ζ or χ blocks.

The matrix λ_{m-2} has the form

$$\begin{pmatrix} \eta & & & & \\ & \delta & & & \\ & & \alpha & & \\ & & & \beta & \\ & & & & \gamma' \end{pmatrix},$$

and the matrix λ_{m-1} has the form

$$\begin{pmatrix} \eta & & \\ & \alpha & \\ & & \beta' \end{pmatrix}.$$

Lemma 2.2. *Let $(Q^\bullet, \phi)$ be a minimal projective resolution of ${}_C\Bbbk$ where the map $Q^i \to Q^{i-1}$ is given as right multiplication by a matrix ϕ_i. Then the matrices ϕ can be chosen to have block form such that all the entries in ϕ_i are elements from the subalgebra generated by the set $\cup_{j=1}^{m+2-i} S_j$.*

Proof. We prove this by induction on i. ϕ_1 can be chosen to be λ_1, which has entries from $V = \cup_{j=1}^{m+1} S_j$. Since the set $\phi_2\phi_1$ must span the space I, ϕ_2 can be chosen to be λ_2, where the blocks have entries of the appropriate form. Now suppose ϕ_i has block form with entries from the subalgebra generated by $\cup_{j=1}^{m+2-i} S_j$. Since the rows of ϕ_{i+1} must annihilate the columns of ϕ_i, ϕ_{i+1} can be chosen to have block form corresponding to the blocks of ϕ_i. Recall that any basis for I is ordered so that only elements of S_j appear on the left of elements of S_{j+1}. Since the entries in ϕ_i contain no elements from $\cup_{j=m+3-i}^{m+1} S_j$, no elements from $\cup_{j=m+2-i}^{m+1} S_j$ will appear in entries of ϕ_{i+1}. Thus the entries in ϕ_{i+1} are from the subalgebra generated by the set $\cup_{j=1}^{m+1-i} S_j$. □

Remark 2.3. The lemma implies that ϕ_{m+1}, if it exists, can only contain elements of S_1 and that there can be no map ϕ_{m+2}. Therefore a minimal resolution of ${}_C\Bbbk$ would have length no more than $m+1$ and so the global dimension of C is at most $m+1$. We will see later that the global dimension of C is exactly m.

Lemma 2.4. *The left annihilators of η, η', λ_m and α are zero.*

Proof. The relations for C make it clear that nothing annihilates n from the left, and consequently η, η' and λ_m cannot be annihilated from the left. Since the entries in α are all from S_2, the annihilator would have to be made from left multiples of n. However p, q and r each appear alone in the first columns of α and n does not annihilate these individually. □

Lemma 2.5. *The rows of γ' generate the left annihilator of x_2.*

Proof. The left annihilator of x_2 can have generators made from sums of elements in S_4, S_3S_4, $S_2S_3S_4$ or $S_1S_2S_3S_4$. It is easy to check that linear combinations of v, w and x_1 are the only elements in the span of S_4 that annihilate x_2. The elements of S_3S_4 span a six dimensional subspace of C_2 with basis $\{sv, sx_1, tv, tw, uv, uw\}$ and these are all left multiples of v, w or x_1. It follows that in C the elements of $S_2S_3S_4$ and $S_1S_2S_3S_4$ are also all left multiples of v, w or x_1. □

Lemma 2.6. *The left annihilator of β is generated by the rows of α, the left annihilator of δ is generated by η, the left annihilator of β' is generated by λ_{m+1}, and the left annihilator of γ' is generated by the rows of β'.*

Proof. We consider an algebra B closely related to C. Let B be the algebra with generators $\{n, p, q, r, s, t, u, v, w, x_1, y_1, a, b\}$ and defining relations $\{np-nq, np-nr, ps-pt, qt-qu, rs-ru, sv-sw, tw-tx_1, uv-ux_1, sv-sy_1, tw-ty_1, ux_1-uy_1, va-vb, wa-wb, x_1a-x_1b\}$. We will resolve $\Bbbk$ as a B-module using maps made from the same blocks as those which appear in the resolution of $\Bbbk$ as a C module.

Let $(R^\bullet, \psi)$ be this sequence of projective B-modules

$$\begin{array}{ccccccccccc} & R^4 & & R^3 & & R^2 & & R^1 & & R^0 & \\ 0 \to & B[-5] & \xrightarrow{\psi_4} & B[-3]^7 & \xrightarrow{\psi_3} & B[-2]^{14} & \xrightarrow{\psi_2} & B[-1]^{13} & \xrightarrow{\psi_1} & B & \to \Bbbk \end{array}$$

The map from B to $\Bbbk$ is the usual augmentation, the map ψ_1 is right multiplication by the transpose of the matrix

$$\begin{pmatrix} n & p & q & r & s & t & u & v & w & x_1 & y_1 & a & b \end{pmatrix},$$

the map ψ_2 is right multiplication by the the matrix

$$\begin{pmatrix} \eta' & & & & \\ & \delta & & & \\ & & \beta & & \\ & & & \gamma' & -\gamma' \end{pmatrix},$$

the map ψ_3 is right multiplication by the the matrix

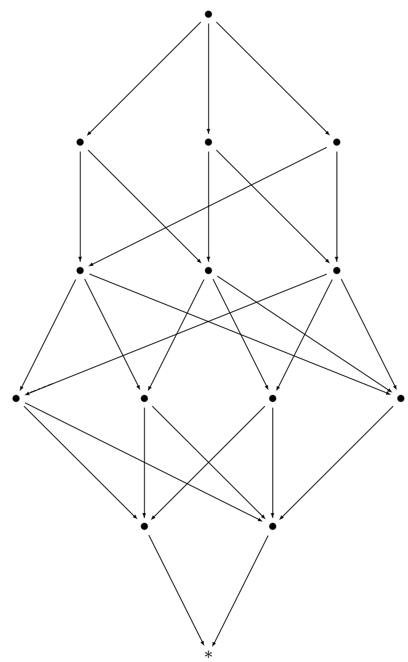
$$\begin{pmatrix} 0 & \eta & & \\ & & \alpha & \\ & & & \beta' \end{pmatrix},$$

and the map ψ_4 is right multiplication by the the matrix

$$\begin{pmatrix} 0 & 0 & 0 & 0 & np & np & -np \end{pmatrix}.$$

Since the product $\psi_i \psi_{i-1}$ is zero in B, $R^\bullet$ is a complex. The list of generators and relations for B ensure that this complex is exact at R^1 and R^0. Our goal is to show that $(R^\bullet, \psi)$ is a minimal projective resolution of ${}_B\Bbbk$.

We observe that B is the associated graded algebra for the splitting algebra (see [7]) corresponding to the layered graph below.



The methods of [7] (see also [6]) show that B has Hilbert series $H_B(g) = (1 - 13g + 14g^2 - 7g^3 + g^5)^{-1}$. Let $P_B(f,g) = \sum_{i,j=0}^{\infty} dim(Ext_B^{ij}(\Bbbk, \Bbbk)) f^i g^j$ be the double Poincaré series for B. From the formula $P_B(-1, g) = H_B^{-1}(g) = 1 - 13g + 14g^2 - 7g^3 + g^5$ we can deduce something about the shape of a minimal projective resolution of the B-module $\Bbbk$. Moreover, the entries in the maps of a minimal projective resolution come from certain sets as in Lemma 2.2. It follows that the resolution must have the form:

$$0 \to B[-5]^{d_3 - d_2} \longrightarrow R^4 \oplus B[-5]^{d_3} \oplus B[-4]^{d_1} \longrightarrow$$

$$R^3 \oplus B[-4]^{d_1} \oplus B[-5]^{d_2} \longrightarrow R^2 \xrightarrow{\psi_2} R^1 \xrightarrow{\psi_1} R^0 \to \Bbbk$$

We will show that $d_1 = d_2 = d_3 = 0$ so that $R^\bullet$ is in fact a resolution of ${}_B\Bbbk$.

Consider first the possibility that $d_1 > 0$. This can only happen if η, α or β' are annihilated from the left by a linear vector. Clearly nothing annihilates η from the left and by Lemma 2.4 nothing annihilates α from the left. Suppose the vector

$$\begin{pmatrix} e_1 & e_2 & e_3 \end{pmatrix}$$

left annihilates β' where each e_i is a sum of elements from S_2. Then

$$\vec{e} = \begin{pmatrix} e_1 & e_2 & e_3 & 0 & 0 & 0 \end{pmatrix}$$

would left annihilate the β which appears in ψ_2. However the Poincaré series for B assures us that the dimension of $Ext_B^{3,3}(\Bbbk,\Bbbk)$ is seven, which means that $\vec{e}$ would be in the span of the rows of α'. Now observe that no nonzero combination of the rows of α' could produce $\vec{e}$. We conclude that $d_1 = 0$.

Since d_1 is zero, $Ext^{3,4}(\Bbbk,\Bbbk) = Ext^{4,4}(\Bbbk,\Bbbk) = 0$. Now observe that the absence of $Ext^{4,4}(\Bbbk,\Bbbk)$ means that $Ext^{5,5}(\Bbbk,\Bbbk)$ must also be zero, so that $d_2 = d_3$. For $Ext^{3,5}(\Bbbk,\Bbbk)$ to be nonzero, the matrix defining the map $R^3 \oplus B[-5]^{d_2} \to R^2$ must contain sums of elements from $S_1S_2S_3$ which annihilate γ'. The elements of $S_1S_2S_3$ span a one dimensional subspace of C_3 with basis $\{nps\}$. Suppose $nps\begin{pmatrix} h_1 & h_2 & h_3 \end{pmatrix}\gamma' =$ for some $h_i \in \Bbbk$. Since in C_4 $npsv = npsw = npsx_1$, we get $h_1 + h_2 + h_3 = 0$, and thus $nps\begin{pmatrix} h_1 & h_2 & h_3 \end{pmatrix}$ is just a row of β' multiplied on the left by np. Therefore the rows of β' generate the left annihilator of γ' and $d_3 = d_2 = 0$, which means $(R^\bullet, \psi)$ is exact.

In general one would not expect information from resolutions over other algebras to be useful in resolving ${}_C\Bbbk$, however the relations of C and B both follow the pattern that the left annihilator of an element of S_i is generated by elements of S_{i-1}, so that information from B is applicable to C. Thus from the fact that $(R^\bullet, \psi)$ is exact, we see that the left annihilator of β is generated by the rows of α, the left annihilator of δ is generated by η, the left annihilator of β' is generated by λ_{m+1}, and the left annihilator of γ' is generated by the rows of β'. □

Theorem 2.7. *For any integer $m \geq 3$ the complex $P^\bullet$ is a minimal projective resolution of the left C-module $\Bbbk$. It follows that the algebra C has global dimension m and $Ext_C^{ij}(\Bbbk,\Bbbk) = 0$ for all $i < j \leq m$. Moreover C is not a Koszul algebra because $Ext_C^{m,m+1}(\Bbbk,\Bbbk) \neq 0$.*

Proof. Direct calculation shows that $\lambda_i\lambda_{i-1} = 0$ for all i so that $P^\bullet$ is a complex. It is clear from the block form of the matrices λ_i that their rows are linearly independent. Since nothing annihilates n from the left, it is clear that $P^\bullet$ is exact at P^m and that none of the λ_i needs an additional row to annihilate η or η'. The complex is exact at P^1 since the product $\lambda_2\lambda_1$ gives the defining relations for C. We will show that $P^\bullet$ is exact elsewhere by examining the component blocks in the matrices λ_i.

The block ϵ appears in λ_1 and is annihilated on the left by the δ which appears in λ_2. For $i > 1$, the column of λ_i containing ϵ has no other nonzero entries. Suppose that for some $d_i \in C$ we have $\begin{pmatrix} d_1 & d_2 & d_3 \end{pmatrix} \epsilon = 0$. Since $\lambda_2\lambda_1$ give a basis for I, it follows that

$$\begin{pmatrix} 0 & 0 & 0 & 0 & d_1 & d_2 & d_3 & 0 & \cdots & 0 \end{pmatrix}$$

is a sum of left multiples of the rows of λ_2. By the block form of λ_2 this means that $\begin{pmatrix} d_1 & d_2 & d_3 \end{pmatrix}$ is a sum of left multiples of the rows of δ. Therefore the rows of δ generate the left annihilator of ϵ. In the same manner, one sees that the rows of ϵ generate the left annihilator of z_1 and that z_i generates the left annihilator of z_{i+1}.

While γ does not appear in λ_1, the matrix $(v, w, x_1, y_1)^t$ does, and is annihilated by β. Any row annihilating γ must also annihilate $(v, w, x_1, y_1)^t$, and hence the rows of β generate the left annihilator of γ. Likewise, since $(x_i, y_i)^t$ appears in λ_1 we see that the rows of γ generate the left annihilator of χ_2 and that the rows of χ_i generate the left annihilator of χ_{i+1}. For $i > 2$, when x_i appears as the only nonzero entry in a column of λ_{m-1-i}, it can be annihilated by at most the rows of χ_{i-1} since χ_{i-1} annihilates the $(x_i, y_i)^t$ in λ_1. Since the second row of χ_{i-1} does not annihilate x_i, we conclude that x_{i-1} generates the left annihilator of x_i.

Lemmas 2.4, 2.5 and 2.6 complete the proof that $P^\bullet$ is exact. Since C is a graded algebra and the maps λ_i are all of degree at least one, this resolution is minimal. □

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