CENTRAL EXTENSIONS OF STEPHENSON'S ALGEBRAS

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Abstract. This paper completes the classification of central extensions of three dimensional Artin-Schelter regular algebras to four dimensional Artin-Schelter regular algebras. Let A be an AS regular algebra of global dimension three and let D be an extension of A by a central graded element z, i.e. $D/\langle z\rangle = A$. If A is generated by elements of degree one, those algebras D which are again AS regular have been classified in [1] and [2]. If A is not generated by elements of degree one, then A falls under a classification due to Stephenson [3, 4]. We classify the AS regular central extensions of Stephenson's algebras by proving that the regularity of D and z is equivalent to the regularity of z in low degree and this is equivalent to easily verifiable conditions on the defining relations for D.

1. Introduction

Artin-Schelter regular (AS regular), graded k-algebras were defined by M. Artin and W. Schelter in [5]. The AS regular algebras of global dimension three which are generated by degree one elements were classified in [5, 6, 7]. In [3] and [4] Stephenson extended this classification and showed that there are AS regular algebras of global dimension three that are not generated by

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elements of degree one. The AS regular algebras of global dimension four have proven more difficult to classify.

One way to find 4-dimensional regular algebras is to look at central extensions of 3-dimensional regular algebras. We call an algebra D with graded central element z such that $A \cong D/\langle z\rangle$ a *central extension* of A. One notices that for any AS regular algebra A of global dimension 3, the central extension $A \otimes_k k[z]$ is an AS regular algebra of global dimension 4. This observation leads to the question: given a central extension D of a 3-dimensional AS regular algebra, is D itself AS regular? In particular, can one determine regularity from the defining relations for D? This question was answered in [1] and [2] for the cases when A is generated by elements of degree one.

While many examples of 4 dimensional AS regular algebras have been studied, most of these are algebras generated by elements of degree one ([1, 8, 9, 10, 11]). The geometry of 3 dimensional AS regular algebras that are not generated by elements of degree one has been investigated in [4] and [12], and it is reasonable to hope that this kind of inquiry could also be carried out for 4 dimensional AS regular algebras that are not generated by elements of degree one.

In this paper we show that theorems analogous to those in [2] hold for central extensions of Stephenson's algebras. In particular, we show that that the regularity of D and of the central element z are equivalent to the regularity of z in low degree, and we show how to determine this from the defining relations for D. Our main results are stated in Theorems 3.7 and 4.5. Given the relations for an algebra A, the fourth condition in our theorems provides a method for producing new examples of 4 dimensional AS regular algebras. Section 5 contains examples of AS regular algebras which can be found using these theorems. These central extensions are all Noetherian domains.

Section 2 contains technical definitions and the notation which will be used throughout the paper. In sections 3 and 4 we determine which central extensions of Stephenson's algebras are AS regular of global dimension 4. In section 3 we consider the algebras with two generators and in section 4 we look at the algebras with three generators.

2. Definitions and Notation

Throughout the paper, k denotes an algebraically closed field of characteristic different from two. All algebras herein are assumed to be finitely generated, connected, $\mathbb{N}$-graded k-algebras. The set of all homogeneous elements of degree n in an algebra B will denoted by B_n. When $n < 0$ let $B_n = 0$.

Definition 2.1. [5, Page 171] A connected $\mathbb{N}$-graded k-algebra B is called Artin-Schelter regular (AS regular) if

(a) the global (homological) dimension of B, denoted $\mathrm{gldim}(B)$, is finite,
(b) the Gelfand-Kirillov dimension of B, denoted $\mathrm{GKdim}(B)$, is finite, and
(c) B is Gorenstein; that is, $\mathrm{Ext}^q_B(k, B) = \delta^q_n k$ for all $q \geq 0$ where $n = \mathrm{gldim}(B)$ and δ^q_n is the Kronecker delta.

Definition 2.2. A homogeneous element z of a graded algebra B is said to be regular if it is neither a left nor a right zero divisor. We say z is n-regular if both left and right multiplication by z are injective on B_n.

Throughout this paper A will be an AS regular algebra of global dimension 3 with fixed generators x_i of degrees μ_i, and D will be a k-algebra with graded central element z of degree α such that $D/\langle z\rangle = A$. If A is generated by A_1, the AS regular central extensions of A have already been classified and we therefore restrict our attention to algebras A which are not generated by elements of degree one. Stephenson [3] has shown that such algebras come in two families. Algebras of type (a, b) have two generators in degrees a and b where $a < b$ and $\gcd(a, b) = 1$. Algebras of type (a, b, c) have three generators in degrees a, b, and c, where $a \leq b \leq c,\ \ a < c$, and $\gcd(a, b, c) = 1$. From [4] and [3] we know that the algebras A are Noetherian domains.

If A is of type (a, b), we will write the generators as $\{x_1, x_2\}$; when A is of type (a, b, c) the generators will be $\{x_1, x_2, x_3\}$. Let $j \in \{2, 3\}$ be the number of generators of A and let $X = (x_1, ..., x_j)^t$. By Proposition 2.5 in [3] we can choose a basis for the vector space of relations, say $\{f_i\}$, so that if $F = (f_1, ..., f_j)^t$, there is a $j \times j$ matrix M with homogeneous entries from $k\langle x_1, ..., x_j\rangle$ such that $MX = F$ and the entries of X^tM generate the ideal $\langle f_1, ..., f_j\rangle$. If A is of type (a, b), set $\delta_1 = deg(f_1) = a + 2b$ and $\delta_2 = deg(f_2) =$

$2a+b$. If A is of type (a,b,c), set $\delta_1 = deg(f_1) = b+c$, $\delta_2 = deg(f_2) = a+c$, and $\delta_3 = deg(f_3) = a+b$.

Let T be the free algebra on the generators of A and on an additional element z of degree α. Let $\tilde{T}$ be a quotient of T wherein the image of z is central, that is $\tilde{T} = T/\langle x_1z - zx_1, ..., x_jz - zx_j\rangle$. The k algebra D will be generated by $\{x_1, ..., x_j, z\}$, where $z \in D_\alpha$ is central. If we write D as $\tilde{T}/I$, then there are homogeneous relations $r_1, ..., r_j \in I$ such that $(r_1, ..., r_j)^t = F + Ez$ where $E = (e_1, ..., e_j)^t$ with homogeneous entries $e_i \in \tilde{T}_{\delta_i - \alpha}$. For notational convenience we will use the same letters to denote elements of the algebras T, $\tilde{T}$, D, and A.

Since z is central and $D/\langle z\rangle = A$, we define the functor π :{graded left D-modules}$\longmapsto${graded left A-modules} by $M \longmapsto M/Mz$.

To each 3 dimensional AS regular algebra which is generated by elements of degree one, we can associate an invertible matrix Q. This matrix is used in [2] to determine which central extensions are AS regular. For Stephenson's algebras, which are not generated by elements in degree one, there is no matrix Q. However, for each of Stephenson's algebras there is a linear function $q : \tilde{T}^j \to \tilde{T}^j$ which will play a role analogous to the matrix Q. Since the entries of X^tM and $\{f_1, ..., f_j\}$ generate the same ideal in $k\langle x_1, ..., x_j\rangle$, each entry of X^tM can be written as a sum of the products of the f_i and certain homogeneous elements from $k\langle x_1, ..., x_j\rangle$. Since $k\langle x_1, x_2, x_3\rangle$ is a subring of $\tilde{T}$, this particular multipication and summation defines a function $q : \tilde{T}^j \to \tilde{T}^j$ such that $q(F) = (X^tM)^t$.

For example, let $A(+)$ be the algebra generated by x, y of degrees 1 and 2 respectively, with two defining relations:

$$f_1 = xy^2 + 2xyx^2 + iy^2x + 2iyx^3$$

$$f_2 = x^2y + yx^2 + y^2$$

This is a global dimension 3 AS regular algebra (cf. [3, 4]). Since $(f_1, f_2)^t = MX$, we see that

$$M = \begin{pmatrix} 2xyx + iy^2 + 2iyx^2 & xy \\ yx & x^2 + y \end{pmatrix}.$$

By comparing MX and X^tM we determine that the function q is given by

$$q\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} iu + 2vx \\ v \end{pmatrix}.$$

We will return to this example in section 5. Just like the matrix Q in [2], the function q will determine which central extensions are in fact AS regular.

Since A is AS regular of global dimension 3, there is an exact sequence $S^\bullet$ of left A (or D) modules of the form:

$$\begin{tikzcd}
 & S^3 & S^2 & S^1 & S^0 & & \\
0 \arrow[r] & A[-\delta_1 - \mu_1] \arrow[r, "X^t"] & \bigoplus_{i=1}^{j} A[-\delta_i] \arrow[r, "M"] & \bigoplus_{i=1}^{j} A[-\mu_i] \arrow[r, "X"] & A \arrow[r] & k \arrow[r] & 0
\end{tikzcd}$$

Throughout this paper a_n will refer to the k vector space dimension of A_n and d_n will be the k vector space dimension of D_n.

Remark 2.3. Notice from the above sequence that A has the Hilbert series

$$H(t) = (1 - \sum_{i=1}^{j} t^{\mu_i} + \sum_{i=1}^{j} t^{\delta_i} - t^{\mu_1+\delta_1})^{-1}$$

and that when $j = 2$

$$a_n = a_{n-\mu_1} + a_{n-\mu_2} - a_{n-\mu_1-2\mu_2} - a_{n-2\mu_1-\mu_2} + a_{n-\mu_1-\delta_1} + \delta_0^n$$

for all n, where δ_0^n is the Kronecker delta. Likewise, when $j = 3$,

$$a_n = \sum_{i=1}^{3} a_{n-\mu_i} - \sum_{i=1}^{3} a_{n-\delta_i} + a_{n-\mu_1-\delta_1} + \delta_0^n$$

for all n. These recurrence relations are needed only for the proofs of Lemmas 3.3 and 4.4.

3. Extensions of Algebras with Two Generators

In this section A will be AS regular of type (a, b). Stephenson has shown that if $1 < a \leq b$, A can be given a new gradation so that the generators have degree one, and therefore he only considers algebras of type $(1, b)$ where $b > 1$. However, even when A can be regraded, it will not always be possible to regrade D, and so we will not assume that a is 1.

Let $X = (x_1, x_2)^t$ and $\mu_3 = \alpha$, $\delta_3 = \mu_1 + \alpha$ and $\delta_4 = \mu_2 + \alpha$. Let $l = \alpha + 2\mu_1 + 2\mu_2$. Let $P^\bullet$ be an augmented sequence of graded, projective left D modules of the form:

$$\begin{tikzcd}
 & P^4 & P^3 & P^2 & P^1 & P^0 & \\
0 \arrow[r] & D[-l] \arrow[r, "\Omega"] & \bigoplus_{i=1}^{3} D[\mu_i - l] \arrow[r, "\gamma"] & \bigoplus_{i=1}^{4} D[-\delta_i] \arrow[r, "\phi"] & \bigoplus_{i=1}^{3} D[-\mu_i] \arrow[r, "\Omega^t"] & D \arrow[r, two heads, "\epsilon"] & k
\end{tikzcd}$$

with the usual graded augmentation map ϵ and matrices

$$\phi = \left(\begin{array}{cc} M & E \\ \begin{matrix} z & 0 \\ 0 & z \end{matrix} & -X \end{array} \right), \gamma = \left(\begin{array}{cc} \begin{matrix} -z & 0 \\ 0 & -z \end{matrix} & M \\ X^t & q(E)^t \end{array} \right), \Omega = (x_1, x_2, z).$$

$P_n^{\bullet}$ will refer to the degree n part of this sequence.

Let $\pi(P^{\bullet})$ be the sequence of D modules:

$$\begin{array}{ccccccccc} \pi(P^4) & & \pi(P^3) & & \pi(P^2) & & \pi(P^1) & & \pi(P^0) \\ A[-l] & \xrightarrow{\bar{\Omega}} & \bigoplus_{i=1}^{3} A[\mu_i - l] & \xrightarrow{\bar{\gamma}} & \bigoplus_{i=1}^{4} A[-\delta_i] & \xrightarrow{\bar{\phi}} & \bigoplus_{i=1}^{3} A[-\mu_i] & \xrightarrow{\bar{\Omega}^t} & A \end{array}$$

with $\bar{\phi} = \pi(\phi)$, $\bar{\Omega} = \pi(\Omega)$ and $\bar{\gamma} = \pi(\gamma)$.

Section 3.1 consists of technical lemmas, which the reader may prefer to skip.

3.1. **Preliminary Lemmas.**

Remark 3.1. If $\alpha = deg(z) > \mu_1 + 2\mu_2$ then D is the polynomial extension $A[z]$ and so is always AS regular. Therefore we consider elements z of degree $\alpha \leq \mu_1 + 2\mu_2$.

Remark 3.2. Let $deg(z) = \alpha$. Since A and D/zD are isomorphic as graded algebras, we see that $d_{n+\alpha} \leq d_n + a_{n+\alpha}$ with equality if and only if z is n-regular.

Lemma 3.3. *Let $deg(z) = \alpha$. If $P^{\bullet}$ is exact in degree $n+\alpha$ and z is m-regular $\forall m < n$, then z is n-regular.*

Proof. By exactness:

$$\begin{aligned} d_{n+\alpha} = {} & d_{n+\alpha-\mu_1} + d_{n+\alpha-\mu_2} + d_n - d_{n+\alpha-2\mu_2-\mu_1} - d_{n+\alpha-\mu_2-2\mu_1} - d_{n-\mu_1} - d_{n-\mu_2} \\ & + d_{n+\alpha-2\mu_1-2\mu_2} + d_{n-2\mu_2-\mu_1} + d_{n-\mu_2-2\mu_1} - d_{n-2\mu_2-2\mu_1} + \delta_0^{n+\alpha} \end{aligned}$$

By 3.2 applied to $d_{n+\alpha-\mu_1}$, $d_{n+\alpha-\mu_2}$, $d_{n+\alpha-2\mu_2-\mu_1}$, $d_{n+\alpha-\mu_2-2\mu_1}$ and $d_{n+\alpha-2\mu_2-2\mu_1}$ and by 2.3 we obtain

$$d_{n+\alpha} = d_n + a_{n+\alpha-\mu_1} + a_{n+\alpha-\mu_2} - a_{n+\alpha-\mu_1-2\mu_2} - a_{n+\alpha-2\mu_1-\mu_2} + a_{n+\alpha-2\mu_1-2\mu_2} + \delta_0^{n+\alpha}$$

$= d_n + a_{n+\alpha}$ as required. □

Let $(r_1, r_2)^t = F + Ez$, $r_3 = x_1 z - z x_1$ and $r_4 = x_2 z - z x_2$. We wish to write the central extension D as $T/\langle r_1, ..., r_4 \rangle$. Any central extension of A can be expressed as T/I where the ideal I must contain $\{r_1, ..., r_4\}$. The following proposition ensures that when D is a 4 dimensional AS regular algebra the set $\{r_1, ..., r_4\}$ actually generates I, and so we may write $D = T/\langle r_1, .., r_4\rangle$.

Proposition 3.4. *Let deg*$(x_1) = a$, *deg*$(x_2) = b$ *and deg*$(z) = \alpha$. *Suppose* $D = T/I$ *is AS regular of global dimension 4 with* $I = \langle r_1, r_2, ..., r_{m-1} = x_1z - zx_1, r_m = x_2z - zx_2\rangle$ *for some* $m \in \mathbb{N}$, *such that* $r_1 \equiv f_1$ *mod* z *and* $r_2 \equiv f_2$ *mod* z *where* f_1 *and* f_2 *are the two relations in* A. *Then there exists a choice of the relations* r_i *so that m=4.*

Proof. We first consider the case that z is not a superfluous generator. If we temporarily disregard shifts in grading, a minimal projective resolution of the trivial left D module k must have the form:

$$0 \longrightarrow D^{n_2} \longrightarrow D^{n_1} \longrightarrow D^m \longrightarrow D^3 \longrightarrow D \twoheadrightarrow k$$

It follows from the Gorenstein condition that the minimal resolution of k must be 'symmetric' so that $n_1 = 3$ and $n_2 = 1$. Now let g be a postive integer so that the following is a minimal graded projective resolution of k:

$$0 \longrightarrow D[-g] \longrightarrow \bigoplus_{i=1}^{3} D[\mu_i - g] \longrightarrow \bigoplus_{i=1}^{m} D[-\delta_i] \longrightarrow \bigoplus_{i=1}^{3} D[-\mu_i] \longrightarrow D \twoheadrightarrow k$$

Let H_D be the Hilbert series of D. From the resolution we have $H_D(t)p(t) = 1$ where $p(t) = t^g - \sum_{i=1}^{3} t^{g-\mu_i} + \sum_{i=1}^{m} t^{\delta_i} - \sum_{i=1}^{3} t^{\mu_i} + 1$. Since D is infinite dimensional, $p(1) = 0$, and so $m = 4$. Also, since A is a homomorphic image of D and GKdim$(A) = 3$, it follows from [13] 8.2.2 that GKdim$(D) \geq 3$, which means that $p'(1) = 0$ and $p''(1) = 0$. Since z is not superfluous in D it is clear that D requires defining relations which are congruent to f_1 and f_2 mod z, and so we may let $\delta_1 = \mu_1 + 2\mu_2$ and $\delta_2 = 2\mu_1 + \mu_2$. Then $0 = p'(1) = 3\mu_1 + 3\mu_2 + \delta_3 + \delta_4 - 2g$. It follows from Gorenstein symmetry that $\delta_3 = g - \mu_1 - 2\mu_2$ and $\delta_4 = g - 2\mu_1 - \mu_2$. Therefore $0 = p''(1) = 8\mu_1^2 + 16\mu_1\mu_2 + 8\mu_2^2 - 4\mu_1 g - 4\mu_2 g + 2g\alpha - 2\alpha^2 = 2(2\mu_1 + 2\mu_2 - \alpha)(2\mu_1 + 2\mu_2 + \alpha - g)$. Since $\alpha < \mu_1 + 2\mu_2$ we conclude that $g = \alpha + 2\mu_1 + 2\mu_2$, $\delta_3 = \alpha + \mu_1$ and $\delta_4 = \alpha + \mu_2$.

Once we know there are four relations of degrees $\mu_1 + 2\mu_2$, $2\mu_1 + \mu_2$, $\mu_1 + \alpha$ and $\mu_2 + \alpha$ we may pick $(r_1, r_2)^t = F + Ez$, $r_3 = x_1z - zx_1$ and $r_4 = x_2z - zx_2$ as generators for I.

Now suppose that z is a superfluous generator, which means $\alpha \in \{\mu_1 + 2\mu_2, 2\mu_1 + \mu_2\}$. In this case a minimal graded projective resolution of the trivial left D module k must have the form:

$$0 \longrightarrow D[-g] \longrightarrow \bigoplus_{i=1}^{2} D[\mu_i - g] \longrightarrow \bigoplus_{i=1}^{m} D[-\delta_i] \longrightarrow \bigoplus_{i=1}^{2} D[-\mu_i] \longrightarrow D \twoheadrightarrow k$$

Here the Hilbert series for D is given by $1/p(t)$ where $p(t) = t^g - \sum_{i=1}^{2} t^{g-\mu_i} + \sum_{i=1}^{m} t^{\delta_i} - \sum_{i=1}^{2} t^{\mu_i} + 1$. Again $p(1) = 0$, and so $m = 2$. If $\alpha = 2\mu_1 + \mu_2$, D must have a relation of degree $\mu_1 + 2\mu_2$, and $p'(1) = p''(1) = 0$ implies $\delta_2 = \alpha + \mu_1$. In this case I is generated by r_1 and r_3. If $\alpha = \mu_1 + 2\mu_2$, D must have a relation of degree $2\mu_1 + \mu_2$, and $p'(1) = p''(1) = 0$ implies $\delta_2 = \alpha + \mu_2$. Here I is generated by r_2 and r_4. In either case, $g = 2\mu_1 + 2\mu_2 + \alpha$ as before.

Whether or not z is a superfluous generator, the set $\{r_1, ..., r_4\}$ will generate I, and we may write $D = T/\langle r_1, ..., r_4\rangle$. □

One consequence of the proof of Theorem 3.7 is that if D is AS regular, then $P^\bullet$ is a resolution of k, although as we saw in the proof of Propostion 3.4 this resolution is minimal only when z is not superfluous. From here on we assume that $D = T/\langle r_1, ..., r_4\rangle$. The proof of the following lemma is identical to that of Lemma 3.5 in [2].

Lemma 3.5. *$P^\bullet$ is exact at P^1, P^0 and k.*

Now we show that if $P^\bullet$ is a resolution of k, then GKdim(D)=4.

Proposition 3.6. *Suppose $D = \tilde{T}/\langle r_1, r_2\rangle$ and ${}_Dk$ has a projective resolution of the form:*

$$0 \longrightarrow D[-g] \xrightarrow{\Omega} \bigoplus_{i=1}^{3} D[\mu_i - g] \xrightarrow{\gamma} \bigoplus_{i=1}^{4} A[-\delta_i] \xrightarrow{\phi} \bigoplus_{i=1}^{3} D[-\mu_i] \xrightarrow{\Omega^t} D \xrightarrow{\epsilon} k$$

Then GKdim$(D) = 4$.

Proof. The resolution implies that D has the Hilbert series

$$H_D(t) = (t^g - \sum_{i=1}^{3} t^{g-\mu_i} + \sum_{i=1}^{4} t^{\delta_i} - \sum_{i=1}^{3} t^{\mu_i} + 1)^{-1}$$
$$= [(t^{\mu_1} - 1)(t^{\mu_2} - 1)(t^{\alpha} - 1)(t^{\mu_1+\mu_2} - 1)]^{-1}.$$

As in [14], we know GKdim is the order of the pole of $H_D(t)$ at 1 and so we have GKdim$(D) = 4$. □

3.2. **Extensions.**

Recall that $\tilde{T} = k\langle x_1, x_2, z\rangle/\langle x_1 z - zx_1, x_2 z - zx_2\rangle$.

Theorem 3.7. *Let $A = D/\langle z\rangle$ be a global dimension 3 AS regular algebra of type (a, b) defined by relations $F = MX$ where $X^t M = q(F)$. Let $z \in D_\alpha$ be central. Write $D = \tilde{T}/\langle r_1, r_2\rangle$ where $(r_1, r_2)^t = F + Ez$ for E in $\tilde{T}^2$. The following are equivalent:*

1. *D is an Artin-Schelter regular algebra (of global dimension 4).*
2. *z is a regular element of D.*
3. *z is a $(2b + 2a - \alpha)$-regular element of D.*
4. *$q(E)^t X = X^t E$ in D.*

Remark 3.8. Note that the degree of $X^t E$ is $2a_1 + 2b - \alpha$ so that if $\alpha > b$ the equality in condition (4) exists in degree below that of the defining relations for D. Consequently, if $\alpha > b$, one can verify the equality $q(E)^t X = X^t E$ in $\tilde{T}$.

Remark 3.9. Since the defining relations for D have degrees $b+2a$ and $2b+a$, the element z is automatically n-regular for all $n \leq b + 2a - \alpha$.

Proof of Theorem 3.7.

(**1**) $\Rightarrow$ (**2**): Since z is central in D and $D/\langle z\rangle$ is Noetherian, D is also Noetherian. By theorem 3.9 in [7], Noetherian AS-regular algebras of global dimension four are domains, and hence z is regular in D.

(**3**) $\Rightarrow$ (**4**): The relations for D are given by $MX + Ez$, hence in D^2 we have $MX = -Ez$. This implies that $X^t MX = -X^t Ez$ and that $q(MX) = -q(Ez) = -q(E)z$, since z is central. But $X^t M = q(MX)^t$, and so $q(MX)^t X = -X^t Ez$ which means $q(E)^t Xz = X^t Ez$. Now $X^t E \in D_{2b+2a-\alpha}$ and z is $(2b + 2a - \alpha)$-regular in D, so z can be canceled to get $q(E)^t X = X^t E$.

(**4**) $\Rightarrow$ (**1**): We prove this in three steps. First we show that $P^\bullet$ is a complex. Then we show that $\pi(P^\bullet)$ is exact at $\pi(P^3)$ and $\pi(P^2)$. Finally, we prove by induction that $P^\bullet$ is exact and z is regular.

Step One: We show that $P^\bullet$ is a complex. By Lemma 3.5, it suffices to see that $\Omega\gamma = 0$ and $\gamma\phi = 0$.

$$\Omega\gamma = (-X^t z + zX^t, X^t M + zq(E)^t)$$

Since z is central, $-X^t z + zX^t = 0$ in D^2. Notice that $X^t M + zq(E)^t = q(MX)^t + q(E)^t z = q(MX + Ez)^t = 0$ in D, and so $\Omega\gamma = 0$. Now consider

$$\gamma\phi = \begin{pmatrix} -zM + Mz & -zE - MX \\ X^t M + q(E)^t z & X^t E - q(E)^t X \end{pmatrix}$$

By condition (4) we have $X^t E - q(E)^t X = 0$ in D, and all other entries are clearly zero. Therefore $\gamma\phi = 0$ in D.

Step Two:

We show that $\pi(P^\bullet)$ is exact at $\pi(P^3)$ and $\pi(P^2)$. Since $P^\bullet$ is a complex, $\pi(P^\bullet)$ is also a complex. Let $(a,b,c) \in \pi(P^3)$ such that $(a,b,c)\bar{\gamma} = (cx_1, cx_2, (a,b)M + cq(E)^t) = (0,0,0,0)$. Since A is a domain, $c = 0$, and so $(a,b)M = (0,0)$. Since $S^\bullet$is exact at S^2, $(a,b) = dX^t$ for some $d \in A$ and so $(a,b,c) = d\bar{\Omega}$. Thus $\pi(P^\bullet)$ is exact at $\pi(P^3)$. To see that $\pi(P^\bullet)$ is exact at $\pi(P^2)$, let $(a,b,c,d) \in \pi(P^2)$ such that

$$(a,b,c,d)\bar{\phi} = ((a,b)M, (a,b)E - cx_1 - dx_2) = (0,0,0)$$

Since $(a,b)M = (0,0)$ and $S^\bullet$ is exact at S^2, $(a,b) = gX^t$ for some $g \in A$. Now $0 = gX^t E - (c,d)X = (gq(E)^t - (c,d))X$. Since $S^\bullet$ is exact at S^1, $gq(E)^t - (c,d) = (h,l)M$ for some $(h,l) \in A^2$, or $(c,d) = gq(E)^t - (h,l)M$ and so $(a,b,c,d) = (-h,-l,g)\bar{\gamma}$. Thus $\pi(P^\bullet)$ is exact at $\pi(P^2)$.

Step Three: We now show by induction that $P^\bullet$is exact and z is regular. By Lemma 3.5 $P^\bullet_n$ is exact at P^0_n and P^1_n $\forall n$, so $P^\bullet_n$ is exact for $n \leq 1$ since $P^i_n = 0$ if $i \geq 2$ and $n \leq 1$. Also, z is clearly m-regular for all $m \leq 0$. Since $\alpha \geq 1$ we have established our base case that $P^\bullet_{2-\alpha}$ is exact and that z is m-regular for all $m < 2 - \alpha$. Now assume inductively that $P^\bullet_n$ is exact and that z is m-regular $\forall m < n$. We will show that $P^\bullet_{n+\alpha}$ is exact and z is n-regular.

Recall that $(\mu_1, \mu_2, \mu_3) = (a, b, \alpha)$, $g = \alpha + 2\mu_1 + 2\mu_2$ and $(\delta_1, \delta_2, \delta_3, \delta_4) = (\mu_1 + 2\mu_2, 2\mu_1 + \mu_2, \alpha + \mu_1, \alpha + \mu_2)$. We have this canonical commutative diagram:

$$\begin{tikzcd}
& P^4 & P^3 & P^2 & P^1 \\
0 \arrow[r] & D[-g] \arrow[r, "\Omega"] \arrow[d, two heads] & \bigoplus_{i=1}^3 D[\mu_i - g] \arrow[r, "\gamma"] \arrow[d, two heads] & \bigoplus_{i=1}^4 D[-\delta_i] \arrow[r, "\phi"] \arrow[d, two heads] & \bigoplus_{i=1}^3 D[-\mu_i] \arrow[d, two heads] \\
& \pi(P^4) \arrow[r, "\bar{\Omega}"] & \pi(P^3) \arrow[r, "\bar{\gamma}"] & \pi(P^2) \arrow[r, "\bar{\phi}"] & \pi(P^1)
\end{tikzcd}$$

(i) The complex $P^\bullet_{n+\alpha}$ is exact at $P^2_{n+\alpha}$: Let u be in $P^2_{n+\alpha}$ such that $u\phi = 0$. Then $\pi(u)\bar{\phi} = 0$ and $\pi(P^\bullet)$ is exact at $\pi(P^2)$ by step two above, so there exists $\bar{v} \in \pi(P^3_{n+\alpha})$ with $\bar{v}\bar{\gamma} = \pi(u)$. Let v be in $P^3_{n+\alpha}$ with $\pi(v) = \bar{v}$. Then $\pi(u - v\gamma) = 0$, so $u = v\gamma + zw$ for some $w \in P^2_n$. Now $0 = u\phi = (v\gamma + zw)\phi = z(w\phi)$, but $w\phi \in P^1_n = D_{n-\mu_1} \oplus D_{n-\mu_2} \oplus D_{n-\alpha}$ and z is m-regular for $m < n$, so $w\phi = 0$. By our inductive hypothesis $P^\bullet_n$ is exact, so there exists $w' \in P^3_n$ such that $w'\gamma = w$, and so $u = (v+zw')\gamma$.

(ii) The complex $P^\bullet_{n+\alpha}$ is exact at $P^3_{n+\alpha}$: Let u be in $P^3_{n+\alpha}$ such that $u\gamma = 0$. Then $\pi(u)\bar{\gamma} = 0$ and $\pi(P^\bullet)$ is exact at $\pi(P^3)$, so there exists $\bar{v} \in \pi(P^4_{n+\alpha})$ with $\bar{v}\bar{\Omega} = \pi(u)$. Let v be in $P^4_{n+\alpha}$ with $\pi(v) = \bar{v}$. Then $\pi(u - v\Omega) = 0$, so $u = v\Omega + zw$ for some $w \in P^3_n$. Now $0 = u\gamma = (v\Omega + zw)\gamma = z(w\gamma)$, but $w\gamma \in P^2_n = D_{n-2\mu_2-\mu_1} \oplus D_{n-\mu_2-2\mu_1} \oplus D_{n-\alpha-\mu_1} \oplus D_{n-\alpha-\mu_2}$ and z is m-regular for $m < n$, so $w\gamma = 0$. By hypothesis $P^\bullet_n$ is exact, so there exists $w' \in P^4_n$ such that $w'\Omega = w$, and so $u = (v + zw')\Omega$.

(iii) The complex $P^\bullet_{n+\alpha}$ is exact at $P^4_{n+\alpha}$: Let u be in $P^4_{n+\alpha}$ such that $u\Omega = (ux_1, ux_2, uz) = (0, 0, 0)$. Since $P^4_{n+\alpha} = D_{n-2\mu_2-2\mu_1}$ and z is $(n - 2\mu_2 - 2\mu_1)$-regular, we have $u = 0$.

Therefore $P^\bullet_{n+\alpha}$ is exact. Since z is m-regular $\forall m < n$, z is n-regular by Lemma 3.3, and so by induction z is regular and $P^\bullet$ is exact. This implies that $\mathrm{gldim}(D) = 4$ and that D is Gorenstein. By proposition 3.6 $\mathrm{GKdim}(D) = 4$, and so D is AS regular. □

4. Extensions of Algebras with Three Generators

Throughout this section A is of type (a, b, c), $X = (x_1, x_2, x_3)^t$ and $\mu_4 = \alpha$. Let $\delta_4 = a + \alpha$, $\delta_5 = b + \alpha$ and $\delta_6 = c + \alpha$, so that δ_4, δ_5 and δ_6 are the degrees of the three commutation relations which make z central. Let $l = \alpha + a + b + c$. Let $P^\bullet$ be an augmented sequence of graded, projective left D modules of the form:

$$\begin{array}{ccccccccccc} & P^4 & & P^3 & & P^2 & & P^1 & & P^0 & \\ 0 \to & D[-l] & \xrightarrow{\Omega} & \bigoplus_{i=1}^4 D[\mu_i - l] & \xrightarrow{\gamma} & \bigoplus_{i=1}^6 D[-\delta_i] & \xrightarrow{\phi} & \bigoplus_{i=1}^4 D[-\mu_i] & \xrightarrow{\Omega^t} & D & \overset{\epsilon}{\twoheadrightarrow} k \end{array}$$

with the usual graded augmentation map ϵ and matrices $\phi = \left(\begin{array}{cccc} & M & & E \\ z & 0 & 0 & \\ 0 & z & 0 & -X \\ 0 & 0 & z & \end{array}\right)$,

$\gamma = \left(\begin{array}{cccc} -z & 0 & 0 & \\ 0 & -z & 0 & M \\ 0 & 0 & -z & \\ & X^t & & q(E)^t \end{array}\right)$, and $\Omega = (x_1, x_2, x_3, z)$. The degree n part of this sequence will be denoted by $P^\bullet_n$.

Let $\pi(P^\bullet)$ be the sequence of D modules:

$$\begin{array}{ccccccccc} \pi(P^4) & & \pi(P^3) & & \pi(P^2) & & \pi(P^1) & & \pi(P^0) \\ A[-l] & \xrightarrow{\bar{\Omega}} & \bigoplus_{i=1}^4 A[\mu_i - l] & \xrightarrow{\bar{\gamma}} & \bigoplus_{i=1}^6 A[-\delta_i] & \xrightarrow{\bar{\phi}} & \bigoplus_{i=1}^4 A[-\mu_i] & \xrightarrow{\bar{\Omega}^t} & A \end{array}$$

with matrices $\bar{\Omega} = (x_1, x_2, x_3, 0)$, $\bar{\gamma} = \left(\begin{array}{cccc} 0 & 0 & 0 & \\ 0 & 0 & 0 & M \\ 0 & 0 & 0 & \\ & X^t & & q(E)^t \end{array}\right)$, and $\bar{\phi} = \left(\begin{array}{cccc} & M & & E \\ 0 & 0 & 0 & \\ 0 & 0 & 0 & -X \\ 0 & 0 & 0 & \end{array}\right)$. Notice $\bar{\phi} = \pi(\phi)$, $\bar{\Omega} = \pi(\Omega)$ and $\bar{\gamma} = \pi(\gamma)$.

4.1. **Preliminary Lemmas.**

Remark 4.1. If $\alpha = deg(z) > b + c$ then D is the polynomial extension $A[z]$ and so is always AS regular. Therefore we consider elements z of degree $\alpha \leq b + c$.

Notice that remark 3.2 and Lemma 3.5 apply to algebras of type (a, b, c) as well as to those of type (a, b).

The following Proposition ensures that if D is an AS regular algebra then we can write $D = T/\langle r_1, ..., r_6\rangle$ where $(r_1, r_2, r_3)^t = F + Ez$, $r_4 = zx_1 - x_1z$, $r_5 = zx_2 - x_2z$ and $r_6 = zx_3 - x_3z$.

Proposition 4.2. *Let $deg(x_i) = \mu_i$ and $deg(z) = \alpha$. Suppose $D = T/I$ is AS regular of global dimension 4 with $I = \langle r_1, ..., r_{m-2} = zx_1 - x_1z, r_{m-1} = zx_2 - x_2z, r_m = zx_3 - x_3z\rangle$, such that $r_i \equiv f_i$ mod z where f_1,f_2 and f_3 are the three relations in A. Then there exists a choice of the relations r_i so that m=6.*

Proof. We first consider that case that z is not a superfluous generator. A minimal projective resolution of the trivial left D module k must have the form:

$$0 \longrightarrow D^{n_2} \longrightarrow D^{n_1} \longrightarrow D^{m} \longrightarrow D^{4} \longrightarrow D \twoheadrightarrow k$$

It follows from the Gorenstein condition that the minimal resolution of k must be 'symmetric' so that $n_1 = 4$ and $n_2 = 1$. Now let g be a postive integer so that the following is a minimal graded projective resolution of the trivial left D module k:

$$0 \longrightarrow D[-g] \longrightarrow \bigoplus_{i=1}^{4} D[\mu_i - g] \longrightarrow \bigoplus_{i=1}^{m} D[-\delta_i] \longrightarrow \bigoplus_{i=1}^{4} D[-\mu_i] \longrightarrow D \twoheadrightarrow k$$

Let H_D be the Hilbert series of D. Then $H_D(t)p(t) = 1$ where $p(t) = t^g - \sum_{i=1}^{4} t^{g-\mu_i} + \sum_{i=1}^{m} t^{\delta_i} - \sum_{i=1}^{4} t^{\mu_i} + 1$. Since D is infinite dimensional, $p(1) = 0$ and so $m = 6$. Since $\mathrm{GKdim}(D) \geq 3$, $p'(1) = 0$ and $p''(1) = 0$. Since z is not superfluous in D it is clear that D requires defining relations which are congruent to f_1, f_2 and f_3 mod z, and so we may let $\delta_1 = b + c$, $\delta_2 = a + c$ and $\delta_3 = a + b$. Then $0 = p'(1) = 2a + 2b + 2c + \delta_4 + \delta_5 + \delta_6 - 3g$. It follows from Gorenstein symmetry that $\delta_4 = g - b - c$, $\delta_5 = g - a - c$ and $\delta_6 = g - a - b$. Therefore $0 = p''(1) = 2a^2 + 4ab + 2b^2 + 4ac + 4bc + 2c^2 - 2\alpha^2 - 2ag - 2bg - 2cg + 2\alpha g = 2(\alpha - a - b - c)(g - \alpha - a - b - c)$. Since $\alpha < b + c$ we conclude that $g = a + b + c + \alpha$, $\delta_4 = \alpha + a$, $\delta_5 = \alpha + b$ and $\delta_5 = \alpha + c$.

Once we know the degrees of the six relations we may pick $(r_1, r_2, r_3)^t = F + Ez$, $r_4 = x_1 z - z x_1$, $r_5 = x_2 z - z x_2$ and $r_6 = x_3 z - z x_3$ as generators for I.

Now suppose that z is a superfluous generator, which means $\alpha \in \{a + b, b + c, a + c\}$. In this case a minimal projective resolution of the trivial left D module k must have the form:

$$0 \longrightarrow D[-g] \longrightarrow \bigoplus_{i=1}^{3} D[\mu_i - g] \longrightarrow \bigoplus_{i=1}^{m} D[-\delta_i] \longrightarrow \bigoplus_{i=1}^{3} D[-\mu_i] \longrightarrow D \twoheadrightarrow k$$

Here the Hilbert series for D is given by $1/p(t)$ where $p(t) = t^g - \sum_{i=1}^{3} t^{g-\mu_i} + \sum_{i=1}^{m} t^{\delta_i} - \sum_{i=1}^{3} t^{\mu_i} + 1$. Again $p(1) = 0$, and so $m = 4$. If $\alpha = a + b$, D must have relations of degree $a + c$ and $b + c$, and the Gorenstein condition together with $p'(1) = p''(1) = 0$ imply $\delta_3 = \alpha + a$, $\delta_4 = \alpha + b$ and $g = a + b + c + \alpha$. In this case I is generated by r_1, r_2, r_4, and r_5. The cases that $\alpha = a+c$ and $\alpha = b+c$ are symmetrical.

Whether or not z is a superfluous generator, the set $\{r_1, ..., r_6\}$ will generate I, and we may write $D = T/\langle r_1, ..., r_6\rangle$. ☐

Now we show that if $P^\bullet$ is a resolution of k, then GKdim(D)=4.

Proposition 4.3. *Let $g = \alpha + a + b + c$ and $D = \tilde{T}/\langle r_1, r_2, r_3\rangle$ and suppose ${}_Dk$ has a projective resolution of the form:*

$$0 \to D[-g] \xrightarrow{\Omega} \bigoplus_{i=1}^{4} D[\mu_i - g] \xrightarrow{\gamma} \bigoplus_{i=1}^{6} D[-\delta_i] \xrightarrow{\phi} \bigoplus_{i=1}^{4} D[-\mu_i] \xrightarrow{\Omega^t} D \xrightarrow{\epsilon} k$$

Then GKdim$(D) = 4$.

Proof. The resolution tells us that $H_D(t) = [(t^a-1)(t^b-1)(t^c-1)(t^\alpha-1)]^{-1}$ is the Hilbert series for D and hence GKdim$(D) = 4$. ☐

Lemma 4.4. *If $P^\bullet_{n+\alpha}$ is exact and z is m-regular $\forall m < n$, then z is n-regular.*

Proof. By exactness: $d_{n+\alpha} = d_n + \sum_{i=1}^{3} d_{n+\alpha-\mu_i} - \sum_{i=1}^{3} d_{n+\alpha-\delta_i} - \sum_{i=4}^{6} d_{n+\alpha-\delta_i} + \sum_{i=1}^{3} d_{n-a-b-c+\mu_i} + d_{n+\alpha-a-b-c} - d_{n-a-b-c} + \delta_0^{n+\alpha}$. By 3.2 applied to $d_{n+\alpha-\mu_i}$, $d_{n+\alpha-\delta_i}$ and $d_{n+\alpha-a-b-c}$ and by 2.3 we get

$$\begin{aligned} d_{n+\alpha} = d_n &+ \sum_{i=1}^{3}(d_{n-\mu_i} + a_{n+\alpha-\mu_i}) - \sum_{i=1}^{3}(d_{n-\delta_i} + a_{n+\alpha-\delta_i}) - \sum_{i=4}^{6} d_{n+\alpha-\delta_i} + \\ &\sum_{i=1}^{3} d_{n-a-b-c+\mu_i} + d_{n-a-b-c} + a_{n+\alpha-a-b-c} - d_{n-a-b-c} + \delta_0^{n+\alpha} \\ = d_n &+ \sum_{i=1}^{3} a_{n+\alpha-\mu_i} - \sum_{i=1}^{3} a_{n+\alpha-\delta_i} + a_{n+\alpha-a-b-c} + \delta_0^{n+\alpha} = d_n + a_{n+\alpha} \end{aligned}$$

as required. ☐

4.2. **Extensions.**

Theorem 4.5. *Let $A = D/\langle z\rangle$ be a global dimension 3 AS regular algebra of type (a, b, c) defined by relations $F = MX$ where $(X^tM)^t = q(F)$. Let $z \in D_\alpha$ be central. Write $D = \tilde{T}/\langle r_1, r_2, r_3\rangle$ where $(r_1, r_2, r_3)^t = F + Ez$ for E in $\tilde{T}^3$. The following are equivalent:*

1. *D is an Artin-Schelter regular algebra (of global dimension 4).*

(2) *z is a regular element of D.*

(3) *z is an $(a+b+c-\alpha)$-regular element of D.*

(4) *$X^tE = q(E)^tX$ in D.*

Remark 4.6. Note that if $\alpha > b+c$ then $E = 0$ and D is a polynomial extension. If $\alpha > c$ then the equality in part (4) holds not only in D but also in $\tilde{T}$. Unless z is a superfluous generator, D is a *quantum polynomial ring in 4 variables* as defined in [12], section 3. If $\alpha \notin \{a+b, b+c, a+c\}$ then z is certainly not superfluous.

Proof of Theorem 4.5.

(1) $\Rightarrow$ **(2)** and **(3)** $\Rightarrow$ **(4)** are the same as in the proof of Theorem 3.7, only here $X^tE \in D_{a+b+c-\alpha}$. **(4)** $\Rightarrow$ **(1)** follows the same three steps as in the proof of Theorem 3.7. In the third step we apply induction to this commutative diagram:

$$\begin{tikzcd}
 & P^4 & P^3 & P^2 & P^1 \\
0 \arrow[r] & D[-g] \arrow[r, "\Omega"] \arrow[d, two heads] & \bigoplus_{i=1}^4 D[\mu_i - g] \arrow[r, "\gamma"] \arrow[d, two heads] & \bigoplus_{i=1}^6 D[-\delta_i] \arrow[r, "\phi"] \arrow[d, two heads] & \bigoplus_{i=1}^4 D[-\mu_i] \arrow[d, two heads] \\
 & \pi(P^4) \arrow[r, "\bar{\Omega}"] & \pi(P^3) \arrow[r, "\bar{\gamma}"] & \pi(P^2) \arrow[r, "\bar{\phi}"] & \pi(P^1)
\end{tikzcd}$$

Recall that $(\mu_1, \mu_2, \mu_3, \mu_3) = (a, b, c, \alpha)$, $g = \alpha + \mu_1 + \mu_2 + \mu_3$, and $(\delta_1, \delta_2, \delta_3, \delta_4, \delta_5, \delta_6) = (b+c, a+c, a+b, \alpha+a, \alpha+b, \alpha+c)$. By Lemma 3.5 $P_n^\bullet$ is exact at P_n^0 and P_n^1 $\forall n$, so $P_n^\bullet$ is exact for $n \leq 1$ since $P_n^i = 0$ if $i \geq 2$ and $n \leq 1$. Also, z is clearly m-regular for all $m \leq 0$. Since $\alpha \geq 1$ we have established our base case that $P_{2-\alpha}^\bullet$ is exact and that z is m-regular for all $m < 2-\alpha$. Now assume inductively that $P_n^\bullet$ is exact and that z is m-regular $\forall m < n$. As before, we conclude that $P_{n+\alpha}^\bullet$ is exact, and since z is m-regular for all $m < n$, z is n-regular by Lemma 4.4. Therefore by induction z is regular and $P^\bullet$ is exact. This implies that $\operatorname{gldim}(D) = 4$ and that D is Gorenstein. By proposition 4.3 $\operatorname{GKdim}(D) = 4$, and so D is AS regular. Λ

5. Examples

Example 5.1. We now return to the algebra $A(+)$ discussed in section 2. Here we consider two seemingly similar central extensions, although in fact only one is AS regular.

Let z have degree 4 and define the algebras D and D' as follows:

$$D = \frac{k\langle x, y, z\rangle}{\langle xz - zx, yz - zy, f_1 + 2xz, f_2 + (1-i)z\rangle}$$

$$D' = \frac{k\langle x, y, z\rangle}{\langle xz - zx, yz - zy, f_1 - xz, f_2 + (1-i)z\rangle}$$

Notice that both D and D' are central extensions of A. We have $E = \begin{pmatrix} 2x \\ 1-i \end{pmatrix}$, and $E' = \begin{pmatrix} -x \\ 1-i \end{pmatrix}$. Recall that the function q is given by

$$q\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} iu + 2vx \\ v \end{pmatrix}.$$

Since $q(E)^t X = 2x^2 + y - iy = X^t E$, part (4) of Theorem 3.7 tells us that D is AS regular and z is not a zero divisor in D. In contrast, $q(E')^t X = 2x^2 - 3ix^2 + y - iy$ and $X^t E' = y - iy - x^2$ so that D' is not AS regular and z is not even 2-regular in D'. In fact, z annihilates x^2 since in $\tilde{T}$,

$$6zx^2 = (i-1)r_1x - (i+1)xr_1 + 2(1+i)r_2x^2 + (i+1)r_2y - (i+1)yr_2$$

and so $zx^2 = 0$ in D'.

Example 5.2. Here we consider the family of quantum polynomial rings discussed in [12], example 6.3. The algebras are generated by elements x, y and w of degrees 1,1, and 2 respectively, with the defining relations:

$$f_1 = wy - iyw - t_0y^3 - t_1xy^2 - t_2x^2y - x^3$$

$$f_2 = wx + ixw + iy^3 + t_0xy^2 - it_1x^2y - t_2x^3$$

$$f_3 = yx - ixy,$$

where i is a square root of -1 and $t_i \in k$. Choose a new basis F' for the defining ideal by taking $f_1' = f_1 + t_2(f_3x - xf_3) + it_2(f_3x + xf_3)$, $f_2' = t_1(f_3x + ixf_3) - t_0yf_3 - if_2$ and $f_3' = f_3$. Our new relations are:

$$f_1' = wy - iyw - t_0y^3 - t_1xy^2 - x^3 + t_2(iyx^2 + ix^2y + yx^2)$$

$$f_2' = xw - iwx + y^3 + t_2ix^3 + t_1yx^2 + t_0(iyxy - ixy^2 - y^2x)$$

Letting $X = (x, y, w)^t$, we write $F' = MX$ to find

$$M = \begin{pmatrix} t_2(i+1)yx - x^2 & w - t_0y^2 + t_2ix^2 - t_1xy & -iy \\ t_1yx - t_0y^2 + t_2ix^2 - iw & y^2 + t_0i(yx - xy) & x \\ y & -ix & 0 \end{pmatrix}.$$

We now compare MX and X^tM to find q. Since

$$(X^tM)^t = \begin{pmatrix} t_2(i+1)xyx - x^3 + t_1y^2x - t_0y^3 + t_2iyx^2 - iyw + wy \\ xw - t_0xy^2 + t_2ix^3 - t_1x^2y + y^3 - t_0iyxy + t_0iy^2x - iwx \\ yx - ixy \end{pmatrix}$$

the function q is given by the formula

$$q\begin{pmatrix} s \\ u \\ v \end{pmatrix} = \begin{pmatrix} s + t_1(ivy + yv) + t_2(xv - vx) \\ u + t_0(1+i)(yv - vy) - t_1(vx + ixv) \\ v \end{pmatrix}.$$

There are many choices for E so that $X^tE = q(E)^tX$. Without placing any restrictions on the scalars t_0, t_1 or t_2, we can construct a family of AS regular central extensions as follows: Let $deg(z) = 1$ and let $g_1(x,z)$ and $g_2(y,z)$ be any homogeneous polynomials of degree 2. Set

$$r_1 = f_1' + (g_1(x,z) + t_3y^2 + t_4yz + t_5xy + (t_5i + t_6 - t_6i)yx)z$$

$$r_2 = f_2' + (g_2(y,z) + t_6x^2 + t_4xz + t_7xy + (t_7i + t_3 - t_3i)yx)z$$

where $t_i \in k$. Now we define the algebras D as

$$D = \frac{k\langle x,y,w,z\rangle}{\langle xz - zx, yz - zy, wz - zw, r_1, r_2, f_3\rangle}$$

Notice that D is a central extension of our original algebra. In this case

$$E = \begin{pmatrix} g_1(x,z) + t_3y^2 + t_4yz + t_5xy + (t_5i + t_6(1-i))yx \\ g_2(y,z) + t_6x^2 + t_4xz + t_7xy + (t_7i + t_3(1-i))yx \\ 0 \end{pmatrix}$$

and one can check that in D the equality $X^tE = q(E)^tX$ holds. In fact, if $t_i = 0$ for $i = 3,5,6,7$ this equality holds in $\tilde{T}$. In all cases, z is a regular element of D and D is an AS regular algebra of global dimension 4.

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References

[1] Le Bruyn, L.; Smith, S. P.; Van den Bergh, M. Central extensions of three dimensional Artin-Schelter regular algebras. Math. Zeitschrift **1996**, *222*, 171-212.

[2] Cassidy, T. Global dimension 4 extensions of Artin-Schelter regular algebras. J. Algebra **1999**, *220*, 225-254.

[3] Stephenson, D.R. Artin-Schelter regular algebras of global dimension three. J. Algebra **1996**, *183* (1), 55-73.

[4] Stephenson, D.R. Algebras associated to elliptic curves. Trans. A.M.S. **1997**, *349* (6), 2317-2340.

[5] Artin, M.; Schelter, W. Graded algebras of global dimension 3. Adv. Math. **1987**, *66*, 171-216.

[6] Artin, M.; Tate, J.; Van den Bergh, M. Some algebras associated to automorphisms of elliptic curves. In *The Grothendieck Festschrift*, 1; Cartier, P., Illusie, L., Katz, N.M., Laumon, G., Manin, Y., Eds.; Birkhäuser: Boston, 1990; 33-85.

[7] Artin, M.; Tate, J.; Van den Bergh, M. Modules over regular algebras of dimension 3, Invent. Math. **1991**, *106*, 335-388.

[8] Shelton, B.; Tingey, C. On Koszul algebras and a new construction of Artin-Schelter regular algebras. J. Algebra **2001**, *241* (2), 789-798.

[9] Smith, S. P.; Stafford, J. T. Regularity of the four-dimensional Sklyanin algebra. Compositio Math. **1992**, *83* (3), 259-289.

[10] Vancliff, M. Quadratic algebras associated with the union of a quadric and a line in $\mathbb{P}^3$. J. Algebra **1994**, *165*, 63-90.

[11] Vancliff, M.; Van Rompay, K. Embedding a quantum nonsingular quadric in a quantum $\mathbb{P}^3$. J. Algebra **1997**, *195* (1), 93-129.

[12] Stephenson, D. R. Quantum planes of weight (1,1,n). J. Algebra **2000**, *225* (1), 70-92.

[13] McConnell, J.; Robson, J. *Non-Commutative Noetherian Rings*; Wiley-Interscience: New York, 1987; 596 pp.

[14] Stephenson, D. R.; Zhang, J. J. Growth of graded Noetherian rings. Proceedings of the AMS **1997**, *125*, 1593-1605.