HOMOGENIZED DOWN-UP ALGEBRAS

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Abstract. This paper studies two homogenizations of the down-up algebras introduced in [1]. We show that in all cases the homogenizing variable is not a zero-divisor, and that when the parameter β is non-zero, the homogenized down-up algebra is a Noetherian domain and a maximal order, and also Artin-Schelter regular, Auslander regular, and Cohen-Macaulay. We show that all homogenized down-up algebras have global dimension 4 and Gelfand-Kirillov dimension 4, and with one exception all homogenized down-up algebras are prime rings. We also exhibit a basis for homogenized down-up algebras and provide a necessary condition for a Noetherian homogenized down-up algebra to be a Hopf algebra.

1. Introduction

The down-up algebra $A(\alpha, \beta, \gamma)$ was introduced by Benkart and Roby in [1] as an associative algebra with generators d and u and defining relations

$$d^2u = \alpha dud + \beta ud^2 + \gamma d$$

$$du^2 = \alpha udu + \beta u^2d + \gamma u$$

where α, β, γ are fixed but arbitrary elements of $\mathbb{C}$. Many well known algebras appear as down-up algebras. Examples include the enveloping algebras of the Heisenberg Lie algebra and the Lie algebra $\mathfrak{sl}_2$. Benkart and Roby conclude

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their paper with a list of open questions concerning down-up algebras. Question (f) on this list is the study of homogenized down-up algebras. These are algebras generated by d, u and t subject to the relations

$$d^2u = \alpha dud + \beta ud^2 + \gamma dt^2,$$

$$du^2 = \alpha udu + \beta u^2d + \gamma ut^2,$$

$$dt = td, \qquad ut = tu.$$

In this paper we study these homogenizations and also homogenizations in which the generator t is of degree two.

In [2] Le Bruyn and Smith homogenize the enveloping algebra of $\mathfrak{sl}_2$ to create an algebra with 4 generators and 6 quadratic relations. Their version of homogenized $U(\mathfrak{sl}_2)$ is a positively graded Noetherian domain, a maximal order, and an Auslander-regular ring of dimension 4. It is also Artin-Schelter regular and satisfies the Cohen-Macaulay property. It is natural to wonder which of these properties are shared by the homogenized down-up algebras.

In section three we establish that in a homogenized down-up algebra the element t is not a zero-divisor, and in the process we determine when a homogenized down-up algebra is Artin-Schelter regular. This tells us when a homogenized down-up algebra is Auslander regular, Cohen-Macaulay and a maximal order. In [3] and [4] it is shown that down-up algebras are Noetherian domains and Auslander regular if and only if $\beta \neq 0$. We show that the same is true for homogenized down-up algebras. Benkart and Roby prove that that down-up algebras have Gelfand-Kirillov dimension 3, and in [3] and [4] it is shown that down-up algebras have global dimension three. We show that homogenized down-up algebras have both global dimension 4 and Gelfand-Kirillov dimension 4.

In [4] and [5] conditions are provided for a down-up algebra to be a prime or primitive ring; in section four we show that homogenized down-up algebras are never primitive and, with one exception, always prime. A monomial basis for down-up algebras is provided in [1]; we exhibit a similar basis for homogenized down-up algebras. In [6] the question of when a Noetherian down-up algebra has a Hopf structure is studied. We consider this question for homogenized down-up algebras and reproduce one of the results from [6].

2. Preliminaries

Throughout the paper, k denotes an uncountable algebraically closed field of characteristic zero. All algebras are assumed to be finitely generated, graded k-algebras. Grading will be denoted by subscripts.

The parameters α, β, γ are from the field k. Down-up algebras will be denoted by A or by $A(\alpha, \beta, \gamma)$. The k-algebra $H^i = H^i(\alpha, \beta, \gamma)$ is generated by $\{d, u, t\}$ where $deg(d) = deg(u) = 1$ and $deg(t) = i \in \{1, 2\}$. H^1 has defining relations:

$$d^2u = \alpha dud + \beta ud^2 + \gamma dt^2, \quad du^2 = \alpha udu + \beta u^2d + \gamma ut^2,$$

$$dt = td, \quad ut = tu.$$

H^2 has defining relations:

$$d^2u = \alpha dud + \beta ud^2 + \gamma dt, \quad du^2 = \alpha udu + \beta u^2d + \gamma ut,$$

$$dt = td, \quad ut = tu.$$

Note that both H^1 and H^2 are connected and $\mathbb{N}$-graded. For notational convenience we will use the same letters to denote elements of the algebras H^1, H^2, and A.

Definition 2.1. A homogeneous element t of a graded algebra A is said to be regular if it is neither a left nor a right zero divisor. We say t is n-regular if both left and right multiplication by t is injective on A_n. We say t is left regular if $at \neq 0$ for all non-zero $a \in A$.

We refer the reader to [7] for definitions of Artin-Schelter regular (AS regular), Auslander regular, and Cohen-Macaulay rings and to [8] for the definitions of Gelfand-Kirillov dimension and maximal order. The Gelfand-Kirillov dimension of an algebra A will be denoted by GKdim(A), and the global (or homological) dimension of A will be denoted by gldim(A). Vector space dimension will be donoted by $dim_k(A)$.

Remark 2.2. It is shown in [1] Theorem 3.1 that the set $\{u^i(du)^jd^k | i, j, k \in \mathbb{N}\}$ is a basis for a down-up algebra over $\mathbb{C}$, and in fact this holds over any field k. It follows that all down-up algebras are infinite dimensional as vector spaces, and that d is left regular in every down-up algebra.

3. Regularity and dimension results

We begin by showing that in all cases the homogenizing element $t \in H^i$ is not a zero-divisor. This is not immediately obvious, as the following example illustrates.

Example 3.1. Consider the k-algebra R generated by d, u and t subject to the relations

$$d^2u = \alpha dud - ud^2 + dt^2, \qquad du^2 = \alpha udu - u^2d + 2ut^2,$$

$$dt = td, \qquad ut = tu.$$

Although the relations for R closely resemble those of a homogenized down-up algebra, the element t is not regular in this case. Notice that in R we have the equality

$$t(dut - udt) = dt^2u - udt^2 =$$

$$d^2u^2 - \alpha dudu + ud^2u - (ud^2u - \alpha udud + u^2d^2) =$$

$$\alpha dudu - du^2d + 2dut^2 - \alpha dudu + ud^2u - (ud^2u - \alpha udud - du^2d + \alpha udud + 2udt^2)$$

$$= 2t(dut - udt).$$

Hence, while $dut - udt$ is clearly nonzero in R, $t(dut - udt) = 0$.

As has been observed in [9], when $\beta \neq 0$ and $\gamma = 0$ down up algebras are AS regular of type S_1 (cf [10]). Consequently, we have the following.

Proposition 3.2. *Let $A = A(\alpha, \beta, \gamma)$ be a down-up algebra with $\beta \neq 0$ and let $H^i = H^i(\alpha, \beta, \gamma)$ be a homogenization of A. Then H^i is a four dimensional AS regular algebra and t is a regular element of H^i.*

Proof. Write the relations for H^1 as

$$u^2d + \frac{\alpha}{\beta}udu - \frac{1}{\beta}du^2 + \frac{\gamma}{\beta}ut^2$$

$$d^2u - \alpha dud - \beta ud^2 - \gamma dt^2$$

where t is a central element of degree 1. Let $X = (d, u)^t$,
$M = \begin{pmatrix} u^2 & \frac{\alpha}{\beta}ud - \frac{1}{\beta}du \\ -\alpha du - \beta ud & d^2 \end{pmatrix}$, $Q = \begin{pmatrix} -\beta & 0 \\ 0 & -1/\beta \end{pmatrix}$, and $E = \begin{pmatrix} \frac{\gamma}{\beta}ut \\ -\gamma dt \end{pmatrix}$.

We now have the relations for H^1 given by $MX + Et$ where $X^tM = (QMX)^t$. Now notice that $E^tQ^tX = X^tE$ so that by Theorem 1.2 in [11], t is a regular element and the algebra H^1 is AS regular with gldim(H^1)=GKdim(H^1)=4.

Similarly, write the relation for H^2 as

$$u^2d + \frac{\alpha}{\beta}udu - \frac{1}{\beta}du^2 + \frac{\gamma}{\beta}ut$$
$$d^2u - \alpha dud - \beta ud^2 - \gamma dt$$

where t is a central element of degree 2. Let X, M and Q be as above, and let $\tilde{E} = \begin{pmatrix} 0 & \frac{\gamma}{\beta} \\ -\gamma & 0 \end{pmatrix}$. We have the relations for H^2 given by $MX + \tilde{E}Xt$ where and $X^tM = (QMX)^t$. Now notice that $\tilde{E}^t = Q\tilde{E}$ so that by Corollary 3.12 in [11], t is a regular element and the algebra H^2 is AS regular with gldim(H^2)=GKdim(H^2)=4. □

Corollary 3.3. *Let $H^i = H^i(\alpha, \beta, \gamma)$ be a homogenized down-up algebra with $\beta \neq 0$. Then H^i is a Noetherian domain, an Auslander regular algebra, and a maximal order. H^i also satisfies the Cohen-Macauley condition.*

Proof. Since t is central and regular, and $H^i/\langle t\rangle \cong A(\alpha, \beta, 0)$ is Noetherian, H^i must be Noetherian. It follows from [12] Theorem 3.9 that H^i is a domain. It follows from [7] that H^i is Auslander regular and Cohen-Macaulay and then from [13] that H^i is a maximal order. □

The fact that H^1 is Auslander regular and Cohen-Macaulay of global dimension four has been obtained independently by Carvalho and Musson [9].

When $\beta = 0$, $H^i = H^i(\alpha, 0, \gamma)$ is neither AS regular, Auslander regular, Cohen-Macaulay, Noetherian, nor a domain. One can easily see from the defining relations that H^i is not a domain. Since H^i has the non-Noetherian $A(\alpha, 0, 0)$ as a homomorphic image, H^i is also not Noetherian. It follows that H^i is not Auslander regular or Cohen-Macaulay. We show below that when $\beta = 0$, H^i is not AS regular. Nonetheless, t is a regular element.

Theorem 3.4. *Let $H^i = H^i(\alpha, 0, \gamma)$. Then*

(1) *t is a regular element of H^i.*
(2) *GKdim(H^i)=gldim(H^i)=4*
(3) *H^i is not AS regular*

Before proving Theorem 3.4 we need to establish several facts. Let $A = A(\alpha, 0, 0)$ and let $S^\bullet$ be the complex of left A (or H^i) modules of the form:

$$\begin{array}{ccccccccccccc} & & S^3 & & S^2 & & S^1 & & S^0 & & & & \\ 0 & \to & A[-4] & \xrightarrow{(d,0)} & A[-3]^2 & \xrightarrow{M} & A[-1]^2 & \xrightarrow{X} & A & \longrightarrow & k & \longrightarrow & 0 \end{array}$$

with maps given by the matrices $X = (d,u)^t$ and $M = \begin{pmatrix} 0 & du - \alpha ud \\ -\alpha du & d^2 \end{pmatrix}$.

Lemma 3.5. *$S^\bullet$ is a projective resolution of the left A module k.*

Proof. Kirkman and Kuzmanovich [4] have shown that all down-up algebras have global dimension 3, and so k must have a projective resolution of the form:

$$0 \;\to\; \bigoplus_{i=1}^{n} A[-\delta_i] \;\longrightarrow\; A[-3]^2 \;\xrightarrow{M}\; A[-1]^2 \;\xrightarrow{X}\; A \;\to\; k$$

From this resolution we can see that A has the Hilbert series $h_A(t) = p(t)^{-1}$ where $p(t) = (1 - 2t + 2t^3 - \sum_{i=1}^{n} t^{\delta_i})$. Since down-up algebras are infinite dimensional, $p(1) = 0$ and consequently $n = 1$. Since down-up algebras have GK-dimension 3, $h_A(t)$ must have a pole of order 3 at 1, and hence $\delta_1 = 4$. Thus a projective resolution of k must have the same shape as $S^\bullet$ and we only need to verify that the maps are correct. One can easily check that $S^\bullet$ is a complex, and clearly $S^\bullet$ is exact at S^0 and S^1. Since d is left regular in A, $S^\bullet$ is exact at S^3. Now by dimension $S^\bullet$ must be exact at S^2. □

Proof of Theorem 3.4. Let $P^\bullet$ be an augmented sequence of graded, projective left H^i modules of the form:

$$\begin{array}{cccccc} & P^4 & & P^3 & & P^2 \\ 0 \to & H^i[-i-4] & \xrightarrow{(d,0,t)} & H^i[-i-3]^2 \oplus H^i[-4] & \xrightarrow{\Lambda} & H^i[-3]^2 \oplus H^i[-i-1]^2 \end{array}$$

$$\begin{array}{cccccc} & P^1 & & P^0 & & \\ \xrightarrow{\phi} & H^i[-1]^2 \oplus H^i[-i] & \xrightarrow{\Omega^t} & H^i & \xrightarrow{\epsilon} & k \end{array}$$

with the usual graded augmentation map ϵ and matrices

$$\Omega = (d,u,t),\; M = \begin{pmatrix} 0 & du - \alpha ud \\ -\alpha du & d^2 \end{pmatrix},\; \Lambda = \begin{pmatrix} -t & 0 & & \\ 0 & -t & & M \\ d & 0 & 0 & -\gamma d t^{2-i} \end{pmatrix},$$

$$\phi = \begin{pmatrix} & M & & E \\ t & 0 & & \\ 0 & t & & -X \end{pmatrix},\; X = \begin{pmatrix} d \\ u \end{pmatrix} \text{ and } E = \begin{pmatrix} -\gamma u t^{2-i} \\ -\gamma d t^{2-i} \end{pmatrix}.$$

Our goal is to established that $P^\bullet$ is a projective resolution k, from which it will be clear that $\mathrm{GKdim}(H^i) = \mathrm{gldim}(H^i) = 4$ and that H^i is not AS regular. Notice that lemmas 3.2 and 3.5 in [11] are valid in this setting. By lemma 3.5 from [11] and the fact that $(d,0,t)\Lambda = 0$ and $\Lambda\phi = 0$, we have that $P^\bullet$ is a complex.

Let $A = A(\alpha, 0, 0)$, let δ_i^2 be the Kronecker delta and let π be the functor from graded left H^i-modules to graded left A-modules via $M \to M/tM$. To show that $P^\bullet$ is exact, we need to examine this related sequence of H^i modules:

$$\begin{array}{ccccc} \pi(P^4) & & \pi(P^3) & & \pi(P^2) \\ A[-i-4] & \xrightarrow{(d,0,0)} & A[-i-3]^2 \oplus A[-4] & \xrightarrow{\bar{\Lambda}} & A[-3]^2 \oplus A[-i-1]^2 \end{array}$$

$$\begin{array}{cc} & \pi(P^1) \\ \xrightarrow{\bar{\phi}} & A[-1]^2 \oplus A[-i] \end{array}$$

with matrices $\bar{\Lambda} = \begin{pmatrix} \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & M \\ \begin{matrix} d & 0 \end{matrix} & \begin{matrix} 0 & -\delta_i^2\gamma d \end{matrix} \end{pmatrix}$ and $\bar{\phi} = \begin{pmatrix} M & \begin{matrix} -\delta_i^2\gamma u \\ -\delta_i^2\gamma d \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & -X \end{pmatrix}$.

First we show that $\pi(P^\bullet)$ is exact at $\pi(P^3)$ and $\pi(P^2)$. Since $P^\bullet$ is a complex, $\pi(P^\bullet)$ is also a complex. Let $(a, b, c) \in \pi(P^3)$ such that $(a, b, c)\bar{\Lambda} = (cd, 0, (a, b)M - c(0, \delta_i^2\gamma d)) = (0, 0, 0, 0)$. Since d is left regular in A, $c = 0$, and so $(a, b)M = (0, 0)$. Since $S^\bullet$ is exact at S^2, $(a, b) = e(d, 0)$ for some $e \in A$ and so $(a, b, c) = e(d, 0, 0)$. Thus $\pi(P^\bullet)$ is exact at $\pi(P^3)$. To see that $\pi(P^\bullet)$ is exact at $\pi(P^2)$, let $(a, b, c, e) \in \pi(P^2)$ such that

$$(a, b, c, e)\bar{\phi} = ((a, b)M, -a\delta_i^2\gamma u - b\delta_i^2\gamma d - cd - eu) = (0, 0, 0)$$

Since $(a, b)M = (0, 0)$ and $S^\bullet$ is exact at S^2, $(a, b) = g(d, 0)$ for some $g \in A$. Now $0 = (c, e + gd\delta_i^2\gamma)X$. Since $S^\bullet$ is exact at S^1, $(c, e + gd\delta_i^2\gamma) = (h, l)M$ for some $(h, l) \in A^2$, and so $(a, b, c, e) = (h, l, g)\bar{\Lambda}$. Thus $\pi(P^\bullet)$ is exact at $\pi(P^2)$.

The proof now proceeds by induction as in step three of the proof of Theorem 3.10 in [11]. We conclude that t is regular and that $P^\bullet$ is exact. Since t is regular, it follows from [8], Theorem 7.3.5 that $\mathrm{gldim}(H^i) = 4$. Thus $P^\bullet$ is a projective resolution of the H^i module k and H^i has the Hilbert series $(t-1)^{-3}(t+1)^{-1}(t^i-1)^{-1}$. Given this Hilbert series, it follows from [14] Theorem 12.6.2 that $\mathrm{GKdim}(H^i) = 4$. □

4. Other Results

An affine algebra with a nontrivial center over an uncountable algebraically closed field cannot be primitive, as shown in [5] Proposition 3.2. Since t is a central element in H^i, a homogenized down-up algebra has a non-trivial center and thus cannot be primitive. However, with one exception, homogenized down-up algebras are prime rings.

Theorem 4.1. *$H^i = H^i(\alpha,\beta,\gamma)$ is a prime ring unless $\alpha=\beta=\gamma=0$.*

Proof. If $\alpha=\beta=\gamma=0$ then H^i is not semiprime since $(u^2d)H^i(u^2d)=0$. Thus $H^i(0,0,0)$ is not a prime ring. Now assume that α, β and γ are not all zero. Let A be the down-up algebras $A(\alpha,\beta,\gamma t^i)$ over the field $k(t)$. We will show that any non-zero polynomial $f(t)\in k[t]$ is regular in H^i, so that we can embed H^i in A. Since by Theorems 3.2 and 3.4 t is a regular element in H, so is t^n for all $n\in\mathbb{N}$. Write $f(t)$ as $\sum_{j=0}^{n} a_j t^j$ where $a_j\in k$ and $a_n\neq 0$. Suppose $f(t)x=0$ for some $x\in H^i$. Let x_m be the highest degree term in x. Since H^i is graded we must have $a_n t^n x_m = 0$. But $a_n\neq 0$ and t^n is regular, so $x_m=0$ and thus $x=0$.

We now embed H^i in A and suppose that $xH^iy=0$ for some $x,y\in H^i$. If $a\in A$ then there exists $f(t)\in k[t]$ such that $af(t)\in H^i$. Therefore $xayf(t)=xaf(t)y=0$, and since $f(t)$ is regular in A, we have $xay=0$, which means $xAy=0$. By [4] Theorem 3.2, A is a prime ring, which implies either $x=0$ or $y=0$. Thus H^i is also prime. □

The fact that H^i can be embedded in a down-up algebra over $k(t)$ is also used to prove the following.

Lemma 4.2. *The set $\mathcal{B}=\{t^ju^k(du)^ld^m|j,k,l,m\in\mathbb{N}\}$ forms a basis for the homogenized down-up algebra $H^i=H^i(\alpha,\beta,\gamma)$.*

Proof. As in the proof of Theorem 4.1 we embed H^i in the down-up algebra $A=A(\alpha,\beta,\gamma t^i)$ over the field $k(t)$. We first show that the set $\mathcal{B}$ is linearly independent in H^i. Suppose that $\sum_{j,k,l,m} a_{j,k,l,m}t^ju^k(du)^ld^m=0$ where $a_{j,k,l,m}\in k$. Since H^i embeds in A, this equality also holds in A. Since $\{u^k(du)^ld^m|k,l,m\in\mathbb{N}\}$ forms a basis for A over $k(t)$, we have $\sum_j a_{j,k,l,m}t^j=0$ for all fixed k,l,m, and thus each $a_{j,k,l,m}=0$.

We show that $\mathcal{B}$ spans H^i by comparing the dimensions of the graded pieces of $\mathcal{B}$ and H^i. Let $D=H^i/\langle t\rangle$ and notice that $D\cong A(\alpha,\beta,0)$ over the field k, so that $\{u^k(du)^ld^m|k,l,m\in\mathbb{N}\}$ forms a basis for D. First consider the case where the degree of t is 1. Let $\mathcal{B}_n=\{t^ju^k(du)^ld^m|j+k+2l+m=n\}$. Then

$$dim_k(\mathcal{B}_n)=\sum_{j=0}^{n} dim_k\{u^k(du)^ld^m|k+2l+m=n-j\}=\sum_{j=0}^{n} dim_k D_{n-j}.$$

Since t is a regular element in H^1, $\sum_{j=0}^{n} dim_k D_{n-j} = dim_k H_n^1$ and so $\mathcal{B}$ spans H^1.

Likewise, when the degree of t is 2 we let $\mathcal{B}_n = \{t^j u^k (du)^l d^m | 2j+k+2l+m = n\}$ and calculate that

$$\begin{aligned} dim_k(\mathcal{B}_n) &= \sum_{j=0}^{\lfloor n/2 \rfloor} dim_k\{u^k(du)^l d^m | k+2l+m = n-2j\} \\ &= \sum_{j=0}^{\lfloor n/2 \rfloor} dim_k D_{n-2j} = dim_k H_n^2. \end{aligned}$$

□

In [1] Benkart and Roby ask when a down-up algebra is a Hopf algebra. Since the Hopf structure on $U(\mathfrak{sl}_2) = A(2,-1,1)$ can be extended to the homogenized down-up algebra $H^1(2,-1,1)$ by setting $\Delta(t) = t \otimes 1 + 1 \otimes t$, $\epsilon(t) = 0$ and $S(t) = -t$, it is natural to also ask which homogenized down-up algebras have Hopf structures. In [6], Kirkman and Musson provide necessary conditions for $A(\alpha, \beta, \gamma)$ to have a Hopf structure. In particular, they show that if a Noetherian down-up algebra $A(\alpha, \beta, \gamma)$ is a Hopf algebra, then $\alpha + \beta = 1$. The same is true for H^1.

Theorem 4.3. *Let $H = H^1(\alpha, \beta, \gamma)$ be a homogenized Noetherian down-up algebra. If H is a Hopf algebra, then $\alpha + \beta = 1$.*

Proof. Let $\alpha + \beta \neq 1$ and for a contradiction assume H is a Hopf algebra. Let I be the ideal of H generated by $du - ud$ so that H/I is a commutative algebra. Note that H/I is isomorphic to $R = k[a,b,c]/\langle ab^2 - \lambda bc^2, a^2b - \lambda ac^2 \rangle$ where a, b and c are the images of d, u and t, and $\lambda = (1-\alpha-\beta)^{-1}\gamma \in k$. By [6] if H is a Hopf algebra so is R. The two ideals $M_0 = \langle a, b, c \rangle$ and $M_1 = \langle a-1, b, c \rangle$ of R are each of codimension 1. By [6] proposition 2.3 we should have $\mathrm{Ext}_R^1(R/M_0, R/M_0) \cong \mathrm{Ext}_R^1(R/M_1, R/M_1)$. Since the vector space dimensions of $\mathrm{Ext}_R^1(R/M_i, R/M_i)$ and M_i/M_i^2 are equal, we have $\dim_k(\mathrm{Ext}_R^1(R/M_0, R/M_0)) = 3$. Notice that in R we have

$$b = (a-1)^2(b+2ab) - c^2(2\lambda a^2 - 3\lambda a) \in M_1^2$$

and hence $\dim_k(M_1/M_1^2)) < 3$, which means R and H cannot be Hopf algebras.

□

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