

BASIC PROPERTIES OF GENERALIZED DOWN-UP ALGEBRAS

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ABSTRACT. We introduce a large class of infinite dimensional associative algebras which generalize down-up algebras. Let K be a field and fix $f \in K[x]$ and $r, s, \gamma \in K$. Define $L = L(f, r, s, \gamma)$ to be the algebra generated by d, u and h with defining relations:

$$[d, h]_r + \gamma d = 0, \quad [h, u]_r + \gamma u = 0, \quad [d, u]_s + f(h) = 0.$$

Included in this family are Smith's class of algebras similar to $U(sl_2)$, Le Bruyn's conformal sl_2 enveloping algebras and the algebras studied by Rueda. The algebras L have Gelfand-Kirillov dimension 3 and are Noetherian domains if and only if $rs \neq 0$. We calculate the global dimension of L and, for $rs \neq 0$, classify the simple *weight* modules for L , including all finite dimensional simple modules. Simple weight modules need not be classical highest weight modules.

Key words and phrases. Down-up algebras, Generalized Weyl algebras, weight modules.

1. INTRODUCTION

Let K be an algebraically closed field of characteristic zero and fix scalars $r, s, \gamma \in K$ and a polynomial $f \in K[x]$. For $\lambda \in K$, we use the notation $[a, b]_\lambda = ab - \lambda ba$. Define $L = L(f, r, s, \gamma)$ to be the algebra generated by d, u and h with defining relations:

$$[d, h]_r + \gamma d = 0, \quad [h, u]_r + \gamma u = 0, \quad [d, u]_s + f(h) = 0. \quad (1)$$

Remark 1.1. L is isomorphic to its opposite ring L^{op} via the map $d \rightarrow u^{op}$, $u \rightarrow d^{op}$ and $h \rightarrow h^{op}$. Furthermore, for nonzero $a \in K$, $L(f, r, s, \gamma) \cong L(af, r, s, \gamma)$ via $d \rightarrow ad, u \rightarrow u$ and $h \rightarrow h$, and so we will often assume f is a monic polynomial.

This family of algebras encompasses many previously studied algebras.

Example 1.2. The down-up algebra $A(\alpha, \beta, \gamma)$ was introduced by Benkart and Roby in [4] as an associative algebra with generators d and u and defining relations

$$\begin{aligned} d^2u &= \alpha dud + \beta ud^2 + \gamma d \\ du^2 &= \alpha udu + \beta u^2d + \gamma u \end{aligned}$$

where α, β, γ are fixed but arbitrary elements of a field K . Down up algebras have been subsequently studied in [14], [6], [12], [13] and other papers. If $A = A(\alpha, \beta, \gamma)$ is a down-up algebra, then $A \cong L(x, r, s, \gamma)$ where r and s are the roots of $\alpha x^2 + \beta x + \gamma = 0$, i.e. $\alpha = r + s$ and $\beta = -rs$. When $\deg(f) = 1$, then L is the down-up algebra $A(r + s, -rs, ab - arb - a\gamma)$, where $f(x) = ax + b$ for $a, b \in K$ with $a \neq 0$.

Example 1.3. In [18] Smith introduced a class of algebras similar to $U(sl_2)$ which have been subsequently studied in [9]. All of Smith's algebras appear as $L(f, 1, 1, 1)$ for various $f \in K[x]$.

Example 1.4. In [21] Witten introduced a seven parameter family of deformations of $U(sl_2)$. These are K -algebras generated by T_-, T_0 and T_+ with relations

$$T_-T_0 + uT_0T_- + u'T_- = 0,$$

$$T_+T_0 + wT_0T_+ + w'T_+ = 0,$$

$$T_-T_+ + vT_+T_- + v'T_0^2 + v''T_0 = 0,$$

where u, u', w, w', v, v' and v'' are in K . All of the down-up algebras except $A(0, 0, \gamma)$ appear in Witten's family of deformations, as do all of Smith's algebras for which $\deg(f) = 2$ and $f(0) = 0$.

Our algebras L appear in Witten's family of deformations if $\deg(f) = 2$, $f(0) = 0$, and $r \neq 0$, or if $\deg(f) = 1$ and r and s are not both zero. Conversely, any of Witten's deformations for which $u = w^{-1}$ and $u' = w'w^{-1}$ will coincide with our algebra $L(v'x^2 + v''x, -w^{-1}, -v, w'w^{-1})$.

Example 1.5. In [15] Le Bruyn studies a class of algebras similar to $U(sl_2)$ which he calls conformal sl_2 enveloping algebras. All of the conformal sl_2 enveloping algebras appear as $L(bx^2 + x, r, s, \gamma)$ for various $b \in K$ and $rs \neq 0$.

Example 1.6. In [17] Rueda studies a class of algebras similar to $U(sl_2)$ which include all of Smith's algebras as well as some of the down-up algebras. All of Rueda's algebras appear as $L(f, 1, s, 1)$ for various $f \in K[x]$ and $s \neq 0$. In section 3 we will show that Rueda's algebras have global dimension 3, unless f is a nonzero constant and either $s = 1$ or s is not a root of unity, in which case the global dimension is 2.

Example 1.7. If $\gamma = 0$, $rs \neq 0$ and f is a monomial of degree ≥ 1 , then L is a three dimensional Artin-Schelter regular algebra. If $\deg(f) = 1$, L is of type S_1 as classified in [1], and if $\deg(f) = 2$, L is of type S'_1 . When $\deg(f) = n + 1 > 2$, L is a quantum planes of type $(1, 1, n)$ as studied by Stephenson in [19] and [20].

In section two we show that many of the basic properties of the algebras in the above examples hold for the entire family of algebras L . In section 3, Theorem 3.1, we determine the global dimension of the algebras L . In section 4 we introduce the notion of weight modules and give a classification of all simple weight modules, including all finite dimensional modules, under the hypothesis that L is Noetherian. This is Theorem 4.10.

2. BASIC PROPERTIES OF L

In this section we show that the algebras L have many of the same properties as down-up algebras. Most of these preliminary results are consequences of the existence of a canonical basis.

Theorem 2.1. *The set $\{u^i h^j d^l \mid (i, j, l) \in \mathbb{N}^3\}$ is a basis for $L(f, r, s, \gamma)$.*

Proof. It is clear from the defining relations for L that dh , hu and du can be generated by this set of monomials, and hence the set spans L . We will use Bergman's diamond lemma (see [5]) to show that the set $\{u^i h^j d^l \mid (i, j, l) \in \mathbb{N}^3\}$ is linearly independent over K . We order the monomials in the free algebra $K\langle d, u, h \rangle$ using the method that Stafford suggested for Smith's algebras ([18]). Assign degrees to the generators by $\deg(d) = \deg(u) = \deg(f)$ and $\deg(h) = 1$. Order the monomials in d, u and h first by degree and then lexicographically with $u < h < d$. The only ambiguity, dhu , is resolvable relative to our ordering since

$$(rhd - \gamma d)u - d(ruh - \gamma u) = rhdu - rduh =$$

$$rh([d, u]_s + f(h)) - r([d, u]_s + f(h))h - rsu([d, h]_r + \gamma d) + rs([h, u]_r + \gamma u)d,$$

and hdu , duh , udh and hud are all $< dhu$. Therefore, by the diamond lemma, $\{u^i h^j d^l \mid (i, j, l) \in \mathbb{N}^3\}$ is a basis for L . \square

Remark 2.2. If we assume $rs \neq 0$, then the argument used above can be used to show that any of the following sets of monomials form bases of L : $\{u^i d^l h^j \mid (i, j, l) \in \mathbb{N}^3\}$, $\{d^l u^i h^j \mid (i, j, l) \in \mathbb{N}^3\}$ and

$$\{u^i (ud)^l h^j, d^i (ud)^l h^j, (ud)^l h^j \mid (i, j, l) \in \mathbb{N}^3, i > 0\}.$$

The last of these bases will be used to analyze weight modules in section 4.

Throughout, let $n = \max\{\deg(f), 1\}$. For $m \in \mathbb{N}$, let $V_m = V_m(L)$ be the subspace of L spanned by the set of monomials $\{u^i h^j d^k \mid i + j + nk \leq m\}$. Note that $u, h \in V_1 \setminus V_0$, but $d \in V_n \setminus V_{n-1}$. From the theorem it is clear that V_m forms a filtration of L and we denote the associated graded algebra by grL .

For the moment, let $M = L(0, r, s, 0)$, with the generators u', h', d' in place of u, h, d respectively. Clearly M is a graded ring, but to make

sense of the following corollary we want to give M the non-standard grading in which $\deg(u') = \deg(h') = 1$ and $\deg(d') = n$.

Corollary 2.3. *The associated graded ring of L , grL , is isomorphic to $M = L(0, r, s, 0)$, with the grading as above.*

Proof. Let \bar{u}, \bar{h} and \bar{d} be the images of u, h and d respectively in grL . These elements satisfy the relations $[\bar{d}, \bar{h}]_r = 0$, $[\bar{h}, \bar{u}]_r = 0$ and $[\bar{d}, \bar{u}]_s = 0$, just like their counterparts u', h', d' in M . It follows from the theorem that \bar{u}, \bar{h} and \bar{d} generate grL , yielding a graded epimorphism $M \rightarrow grL$. But again by the theorem, the dimensions of the two algebras are the same in every degree, so they must be isomorphic. \square

Corollary 2.4. *$L(f, r, s, \gamma)$ has Gelfand-Kirillov dimension three.*

Proof. Since grL is affine, we have $GK(L) = GK(grL) = GK(M)$. We can forget the nonstandard grading on M and instead compute $GK(M)$ from the standard filtration $F_m = \text{span}\{(u')^i(h')^j(d')^k \mid i + j + k \leq m\}$. By the theorem, these monomials are independent and so the Gelfand-Kirillov dimension is 3. \square

Proposition 2.5. *If $rs \neq 0$ then L is a Noetherian domain.*

Proof. We express L as an iterated Ore extension. Let R be the K -algebra generated by $\{h, d\}$ with relation $[d, h]_r + \gamma d = 0$. Let τ be the automorphism of $K[h]$ given by $\tau(h) = (h + \gamma)/r$. Then R is isomorphic to the Ore extension $K[h][d; \tau]$. Now we define an automorphism σ of R by $\sigma(d) = sd$ and $\sigma(h) = rh - \gamma$. Define a linear map $\delta : R \rightarrow R$ by $\delta(d) = -f(h)$ and $\delta(h) = 0$. Observe that δ is a σ -derivation and that L is isomorphic to the Ore extension $R[u; \sigma, \delta]$. Since $K[h]$ is a Noetherian domain and σ is an automorphism, it follows from [16], Theorem 1.2.9, that R and L are Noetherian domains. \square

Proposition 2.6. *If $rs = 0$ then L is not a domain and is neither left nor right Noetherian.*

Proof. Clearly L is not a domain if $r = 0$, since then $d(h + \gamma) = 0$. If $r \neq 0$ and $s = 0$, then $d(f((h + \gamma)/r) + ud) = f(h)d + dud = (f(h) + du)d = 0$. However, $f((h + \gamma)/r) + ud$ is nonzero by Theorem

2.1. For the Noetherian question we need only consider Noetherian on one side and we follow the approach used by Kirkman, Musson and Passman [14] for down-up algebras. Suppose $r = 0$ so that $(h + \gamma)u = 0$. For each $n \geq 0$, we define a right ideal I_n of L by

$$I_n = \sum_{i=0}^n u^i (h + \gamma) L$$

and use our canonical basis to notice that

$$I_n = \sum_{i=0}^n \sum_{j,k=0}^{\infty} K u^i (h + \gamma) h^j d^k.$$

This makes it clear that $u^{n+1}(h + \gamma) \notin I_n$ and hence the chain $I_0 \subset I_1 \subset I_2 \subset \dots$ has proper containments, i.e. L is not right Noetherian. Now suppose $r \neq 0$ and $s = 0$. Let $b = (h + \gamma)/r$ and for each $n \geq 0$, define a left ideal J_n by

$$J_n = \sum_{i=0}^n L(f(b) + ud)d^i.$$

Then, since $d(f(b) + ud) = (f(h) + du)d = 0$ we have, as above,

$$J_n = \sum_{i=0}^n \sum_{j,k=0}^{\infty} K u^j h^k (f(b) + ud)d^i.$$

Since $f(h) = -du$ and $hu = ruh - \gamma u$, no element of J_n can contain the monomial $h^j d^{n+1}$ for any j , and consequently $(f(b) + ud)d^{n+1} \notin J_n$. Again we have an infinite proper chain of (left) ideals and therefore L is not left Noetherian. \square

Generalized Weyl algebras were defined by Bavula (see e.g. [3]). When L is Noetherian, we can present L as a generalized Weyl algebra.

Lemma 2.7. *If $rs \neq 0$ then L is a generalized Weyl algebra.*

Proof. Let D be the commutative polynomial algebra $K[h, a]$ and let σ be the automorphism of D defined by $\sigma(h) = rh - \gamma$ and $\sigma(a) = sa - f(h)$. Adjoin the variables d and u with the rules:

$$du = \sigma(a), \quad ud = a,$$

and for all $x \in D$

$$dx = \sigma(x)d \quad \text{and} \quad ux = \sigma^{-1}(x)u.$$

\square

Generalized Weyl algebras are closely related to ambiskew polynomial rings, [11], so it is no surprise that L can also be presented as an ambiskew polynomial ring. If L is Noetherian, define σ to be the automorphism of $K[h]$ given by $\sigma(h) = rh - \gamma$ and extend σ to $K[h][u; \sigma^{-1}]$ by setting $\sigma(u) = su$. Then $L(f, r, s, \gamma)$ is isomorphic to the ambiskew polynomial ring $R(K[h], \sigma, -f(h), s)$.

Jordan calls the ambiskew polynomial ring $L = R(K[h], \sigma, -f(h), s)$ *conformal* if there is an $a(x) \in K[x]$ with $f(x) = sa(x) - a(rx - \gamma)$. If L is conformal and $s \neq 0$, then the element $z := du - \sigma(a) = s(ud - a)$ is normal in L .

Lemma 2.8. *If $s \neq r^i$ for $1 \leq i \leq n$, then L is conformal.*

Proof. Let $n = \deg(f)$. For $1 \leq i \leq n$ define $\binom{x}{i} := x(x-1)(x-2)\dots(x-i+1)$ and set $p_i = s\binom{x}{i} - \binom{rx-\gamma}{i}$. Since $s \neq r^i$, p_i is a polynomial of degree i and so the set $\{p_i | 0 \leq i \leq n\}$ is a basis for polynomials of degree n or less. It follows that $f = \sum_{i=0}^n a_i p_i$ for some $a_i \in K$, and so we define $a = \sum_{i=0}^n a_i \binom{x}{i}$. \square

Remark 2.9. Even if $s = r^i$, L may be conformal. It follows from [10] Lemma 2.1 that for conformal L with $rs\gamma \neq 0$, the center of L is K if s is not a root of unity, and is $K[(ud - a(h))^m]$ if $s^m = 1$.

In section 7 of [11], Jordan extends the definition of ambiskew polynomial rings by allowing σ to be any endomorphism. Under this extended definition, the non-Noetherian algebras L are also ambiskew polynomial rings. In particular, with σ and δ as in Proposition 2.5, we have $L \cong ([\sigma; u]K[h])[d; \sigma, \delta]$.

We end this section with an observation relating the algebras L to Artin-Schelter regular algebras of global dimension 4. This observation will not be used in later sections. If γ is nonzero, then L is not an \mathbb{N} -graded algebra, at least not in any particularly useful way. However, we can homogenize L by introducing a new variable t as follows. Recall that $n = \max\{1, \deg(f)\}$ and $f(x)$ is monic. Write $f(x) = \sum_0^m a_i x^i$ where $a_i \in k$. We define $f(x, t) = \sum_0^m a_i x^i t^{n+1-i}$. Note that $\deg(f(x, t)) = n + 1$ as long as $f \neq 0$. If $f = 0$, then $f(x, t) = 0$.

We define Homogenized L , denoted $L_H = L_H(f, r, s, \gamma)$, to be the algebra generated by $\{d, u, h, t\}$ subject to the relations

$$\begin{aligned} [d, h]_r + \gamma dt = 0, \quad [h, u]_r + \gamma ut = 0, \quad [d, u]_s + f(h, t) = 0, \\ [d, t] = 0, \quad [u, t] = 0, \quad [h, t] = 0. \end{aligned}$$

L_H is then a graded algebra where d, u, h and t have degrees $n, 1, 1$ and 1 respectively. Notice that L is a homomorphic image of L_H . The reader is referred to [7] or [19] for the definition of an Artin-Schelter regular algebra.

Theorem 2.10. *If $rs \neq 0$, L_H is a Noetherian domain and an Artin-Schelter regular algebra of global dimension 4.*

Proof. Let C be the algebra $L(0, r, s, 0)$. Let R be the subalgebra of C generated by h and d , and notice that R is AS-regular of global dimension two. Define a graded automorphism of R by $\theta(h) = r^{-1}h$ and $\theta(d) = s^{-1}d$. Since C is the Ore extension $R[u; \theta]$, it follows from proposition 3.12 in [19] that C is AS-regular of dimension three.

Since $C \cong L_H/\langle t \rangle$, L_H is a central extension of C , and so we will use Theorem 4.5 from [7] to show that L_H is also AS-regular. We write the relations for C as

$$uh - r^{-1}hu, \quad rshd - sdh, \quad du - sud.$$

Let $X = (d, u, h)^t$, and let $M = \begin{pmatrix} 0 & -r^{-1}h & u \\ rsh & 0 & -sd \\ -su & d & 0 \end{pmatrix}$, so that the relations for C are given by MX and also $X^tM = q(MX)$ where

$$q \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} rsa \\ (rs)^{-1}b \\ c \end{pmatrix}.$$

Now the defining relations for L_H are given by $MX + Et$ where t is a central element of degree 1 and $E = \begin{pmatrix} -\gamma r^{-1}u \\ -s\gamma d \\ \sum_0^n a_i h^i t^{n-i+1} \end{pmatrix}$.

Since $X^tE = q(E)^tX$ it follows from Theorem 4.5 in [7] that t is a regular element and L_H is AS regular with $\text{gldim}(L_H) = \text{GKdim}(L_H) = 4$.

□

Remark 2.11. In this section we have considered various filtrations and associated gradings on L , mostly with an eye towards global dimension computations in the following section. It should also be observed that there is a simple and useful \mathbb{Z} -grading on L generated by giving the generators d , h and u the degrees -1 , 0 and 1 respectively. This grading will be very useful in section 4.

3. GLOBAL DIMENSION

In this section we determine the global dimension of all the algebras L . Since L is isomorphic to its opposite ring, left and right global dimensions are the same for L , and we denote this common dimension by $gldim(L)$. While most of the algebras L have global dimension 3, there are many exceptions.

Theorem 3.1. *Let c be any nonzero scalar and $L = L(f, r, s, \gamma)$. Then the global dimension of L is described in the following table.*

f	r	s	γ	$gldim(L)$	Proposition
$\neq c$	any	any	any	3	3.3
c	$\neq 0, 1$	$\neq 0, 1$	any	3	3.5
c	1	$\neq 0, 1, s^k = 1$	any	3	3.5
c	1	$\neq 0, 1$	0	3	3.5
c	1	$\neq 0, s^k \neq 1$	$\neq 0$	2	3.5
c	$\neq 0$	1	any	2	3.5
c	0	$\neq 0$	any	3	3.7
c	$\neq 0$	0	any	2	3.12
c	0	0	any	1	3.14

The proof of 3.1 is spread out over the section, with specific references as outlined in the table. We begin by bounding the global dimension at 3.

Lemma 3.2. *L has global dimension ≤ 3 .*

Proof. From the previous section we know that L has an associated graded ring M isomorphic to $L(0, r, s, 0)$ with a nonstandard grading. But M can also be given the standard grading with all three generators in degree one. By [8], that ring is Koszul. By 2.1, the Hilbert series of this ring is $H_M(t) = 1/(1-t)^3$ and hence $gldim(M) = 3$. It then follows from [16] 7.6.18 that $gldim(L) \leq 3$. \square

But a simple analysis of the sets of basis elements for the various V_j shows that $w_k - w_{k-n} = k + 1$ for all $k \geq 0$. The formulas follow immediately and hence P^\bullet is a projective resolution of T .

Finally, we can calculate $Ext_L^3(K, K)$. We see that $Hom_L(P^3, T) \cong K$, $Hom_L(P^2, T) \cong K^3$ and the map between them, $(-rsu, -d, rh)$, acts as zero. Hence $Ext_L^3(K, K) \cong K \neq 0$. In light of 3.2, the global dimension of L is 3. \square

It should be noted that the proof of the proposition above is unnecessarily long whenever $rs \neq 0$, since in that case L is a Noetherian iterated Ore extension which admits a one-dimensional module and so one can apply [16], 7.9.18.

Remark 3.4. For the remainder of this section we will assume that $f(x)$ is a non-zero scalar, i.e. $f(x) = c \neq 0$. The isomorphism class of the algebra L is independent of c and we will typically choose c to be 1 or -1 , whichever is more convenient.

Proposition 3.5. *Assume $rs \neq 0$. Then*

- i) If $r \neq 1$ and $s \neq 1$ then $gldim(L(c, r, s, \gamma)) = 3$,*
- ii) If $r = 1$ and s is a root of 1 other than 1, then $gldim(L(c, 1, s, \gamma)) = 3$,*
- iii) If $r = 1$, $s \neq 1$ and $\gamma = 0$, then $gldim(L(c, 1, s, 0)) = 3$,*
- iv) If $r = 1$, s is not a root of 1, and $\gamma \neq 0$, then $gldim(L(c, 1, s, \gamma)) = 2$,*
- v) If $s = 1$, then $gldim(L(c, r, 1, \gamma)) = 2$.*

Proof. We may assume $c = 1$. By lemma 2.7, L is a generalized Weyl algebra over $D = K[h, a]$ where the automorphism σ of A is given by $\sigma(h) = rh - \gamma$ and $\sigma(a) = sa - 1$. Since D has global dimension 2, it follows from lemma 3.2 and from Theorem 2.7 in [3] that L has global dimension 2 or 3.

From Theorem 3.7 in [3], we see that L has global dimension 3 if and only if there exists either a σ -semistable maximal ideal of D or a maximal ideal P and integer $m \geq 1$ such that $a \in P \cap \sigma^m(P)$.

Assume first $r \neq 1$ and $s \neq 1$. Then the ideal $P = (h - \gamma/(r - 1), a - 1/(s - 1))$ is σ -stable. This proves i).

Assume $r = 1$, $s \neq 1$ and that $s^m = 1$ for some $m > 0$. Define maximal ideals P and Q in D by $P = (a, h)$ and $Q = \sigma^m(P)$. Since $\sigma^m(a) = s^m a - \sum_{j=0}^{m-1} s^j = a$, a is in $Q \cap P$. This proves ii).

Assume $r = 1$, $s \neq 1$ and $\gamma = 0$. Then the maximal D -ideal $(h, a - 1/(s-1))$ is σ -stable, and hence L has global dimension 3. This proves iii).

Assume $r = 1$ and s is not a root of 1 and $\gamma \neq 0$. Any maximal ideal of D has the form $P = (h - \mu, a - \lambda)$ for scalars λ and μ . But then for $m \geq 1$, $h - m\gamma - \mu \in \sigma^m(P)$, while $h - m\gamma - \mu \notin P$. Hence no P is σ -semistable. Similarly, since $\sigma^m(a)$ is in $K[a]$, but is never a scalar multiple of a , we can never have a in $P \cap \sigma^m(P)$. This gives us $\text{gldim}(L) = 2$ and proves iv).

Finally, assume $s = 1$. Then $\sigma^m(a) = a - m$ and so, as in the previous paragraph, there can be no σ -semistable maximal ideals P , nor can we have a in $P \cap \sigma^m(P)$. This proves v). \square

It remains only to consider the cases where $r = 0$ or $s = 0$. We begin with the cases where $r = 0$ and $s \neq 0$. We require a technical lemma about annihilators of elements in $L(-1, 0, s, 0)$.

Lemma 3.6. *Assume $s \neq 0$. If $s \neq 1$, let $I_s = \{k \in \mathbb{Z} : s^k = 1\}$. Let $I_1 = \{0\}$. We have the following annihilators in $L = L(-1, 0, s, 0)$: $rann(h) = uL$, $lann(h) = Ld$, $rann(d) = \bigoplus_{k \in I_s} u^k hL$, and $lann(u) = \bigoplus_{k \in I_s} Lhd^k$.*

Proof. In $L = L(-1, 0, s, 0)$ we have the relations $dh = hu = du - sud + c = 0$. It is clear from the relations for L that $uL \subset rann(h)$ and $Ld \subset lann(h)$. It follows easily from Theorem 2.1 that $rann(h) = uL$ and $lann(h) = Ld$. Since $du^m = s^m u^m d + u^{m-1} \sum_{i=1}^m s^{i-1}$, it follows that $\bigoplus_{k \in I_s} u^k hL \subset rann(d)$.

To see $rann(d) \subset \bigoplus_{k \in I_s} u^k hL$, assume $dx = 0$ for some $x \in L$ and proceed by induction on the degree in u of x , $\text{deg}_u(x)$ (which is well defined by 2.1). If $\text{deg}_u(x) = 0$ it follows easily that $x \in hL$ and $hL \subset \bigoplus_{k \in I_s} u^k hL$ since $0 \in I_s$. Now let $\text{deg}_u(x) = m$ and assume the

claim is true for y with $\deg_u(y) < m$. Write x as $\sum_{i=0}^m u^i \alpha_i$ where $\alpha_i \in hL + \sum_j Kd^j$. Since $du^m = s^m u^m d + u^{m-1} \sum_{i=1}^m s^{i-1}$, we have $dx = s^m u^m d\alpha_m + w$ where w , written in the basis from Theorem 2.1, has degree in u less than m . It follows from the basis that $s^m u^m d\alpha_m = 0$, and since $s \neq 0$ and u is right regular we have $d\alpha_m = 0$. Since $\deg_u(\alpha_m) = 0$, $\alpha_m = hz$ for some $z \in hL + \sum_j Kd^j$. Now we have $x = u^m hz + u^{m-1} \alpha_{m-1} + \sum_{i=0}^{m-2} u^i \alpha_i$, and since $dh = 0$, $0 = dx = -u^{m-1} \sum_{i=1}^m s^{i-1} hz + s^{m-1} u^{m-1} d\alpha_{m-1} + y$ where $\deg_u(y) < m-1$. Since the highest terms is u must cancel we conclude that

$$s^{m-1} d\alpha_{m-1} = \sum_{i=1}^m s^{i-1} hz.$$

Since d annihilates h from the left, $s^{m-1} d\alpha_{m-1}$ contains no h terms, which implies that either $z = 0$ or $\sum_{i=1}^m s^{i-1} = 0$. If $z = 0$ then $\deg_u(x) < m$ and so $m \in I_s$ by induction. If $\sum_{i=1}^m s^{i-1} = 0$ then s is an m^{th} root of unity and again $m \in I_s$.

Now, since $m \in I_s$, $du^m hz = 0$ and hence $0 = d(x - u^m hz)$. This means $0 = d \sum_{i=0}^{m-1} u^i \alpha_i$ and $\deg_u(\sum_{i=0}^{m-1} u^i \alpha_i) < m$ so by induction $\sum_{i=0}^{m-1} u^i \alpha_i \in \bigoplus_{k \in I_s} u^k hL$. Thus $x = u^m hz + \sum_{i=0}^{m-1} u^i \alpha_i \in \bigoplus_{k \in I_s} u^k hL$.

A parallel argument shows that $\text{lann}(u) \subset \bigoplus_{k \in I_s} Lhd^k$ \square

Proposition 3.7. *For $s \neq 0$, $L(c, 0, s, \gamma)$ has global dimension 3.*

Proof. As usual, we may assume $c = -1$. Moreover, the change of generators $d \rightarrow d, u \rightarrow u$ and $h \rightarrow h - \gamma$ allows us to assume $\gamma = 0$, i.e. $L = L(-1, 0, s, 0)$. Let I_s be as in Lemma 3.6. It follows from the lemma that the following two sequences of right L modules are exact:

$$0 \longrightarrow uL \longrightarrow L \longrightarrow hL \longrightarrow 0$$

$$0 \longrightarrow \bigoplus_{k \in I_s} u^k hL \longrightarrow L \longrightarrow dL \longrightarrow 0$$

We will show that the first sequence is non-split. For a contradiction, suppose we had a map $\tau : hL \rightarrow L$ with $h\tau(a) = a$ for all $a \in hL$. Then $\tau(h)$ would be in $\text{lann}(u) = \bigoplus_{k \in I_s} Lhd^k$, and therefore $\tau(h) = \sum_{k \in I_s, i, j} \alpha_{ijk} u^i h^{j+1} d^k$ for some $\alpha_{ijk} \in K$. Since $h\tau(h) = h$ we

have $\sum_{k \in I_{s,j}} \alpha_{0jk} h^{j+2} d^k = h$ so that $h(\sum_{k \in I_{s,j}} \alpha_{0jk} h^{j+1} d^k - 1) = 0$ and thus $(\sum_{k \in I_{s,j}} \alpha_{0jk} h^{j+1} d^k - 1) \in \text{rann}(h) = uL$, which is impossible since u is right regular and $\sum_{k \in I_{s,j}} \alpha_{0jk} h^{j+1} d^k - 1$ had degree 0 in u .

Since u is right regular in L , uL is free, and we have established that the projective dimension of hL is 1. Consequently the projective dimension of $\bigoplus_{k \in I_s} u^k hL$ is also at least 1. From the second sequence we see that the projective dimension of dL is at least 2, and hence the projective dimension of L/dL is 3. \square

Next we consider the algebras where $s = 0$ but $r \neq 0$. We require a preliminary lemma that should be well-known, but for which we have no reference.

Lemma 3.8. *Let $S = K\langle d, u \rangle / \langle du - 1 \rangle$. Then S is hereditary.*

Proof. Observe first that the set $\{u^i d^j\}$ forms a basis for S . Let I be the two-sided ideal in S generated by the idempotent $1 - ud$. We wish to establish that as a left S -module I is semisimple and projective. Let D be the right annihilator of d in S and observe that $D = (1 - ud)S$. Since $du = 1$ and $(1 - ud)u = 0$, any element in D can be written as $\sum_{j=0}^m \alpha_j (1 - ud) d^j$ for $\alpha_j \in K$. Then

$$\left(\sum_{j=0}^m \alpha_j (1 - ud) d^j \right) u^m = \alpha_m (1 - ud),$$

and hence D is a simple right S -module. Since $1 - ud$ is an idempotent, D is also projective. Since $d(ud - 1) = 0$ and u is right regular in S it follows that $u^i D \cong D$ and $I = \bigoplus_{i \geq 0} u^i D$. Therefore I is projective and semisimple.

Now let B be any right ideal in S and consider two cases.

Case 1 $I \subset B$. We may assume $I \neq B$ and then B/I is a nonzero right ideal in $S/I \cong K[d, d^{-1}]$. Since $K[d, d^{-1}]$ is a P.I.D. we may write B as $p(d)S + I$ for some $p \in K[d] \subset S$. Without loss of generality, $p(0) = 1$. Since $p(d) \neq 0$ we may write $p(d)$ as $\prod_{i=1}^m (d - \lambda_i)$ with $\lambda_i \neq 0$ for all i . By looking at the highest power of u in $(d - \lambda_i) \sum_{i,j} \alpha_{ij} u^i d^j$

we see that $d - \lambda_i$ is right regular and hence $p(d)$ is right regular, so that $p(d)S \cong S$.

Now we show that $I \subset p(d)S$. Let p have degree m and note that $(p(d) - 1)u \in K + Kd + Kd^2 + \dots + Kd^{m-1}$. From this it follows easily that the vector space $p(d)S + K + Kd + Kd^2 + \dots + Kd^{m-1}$ forms a right ideal of S . This ideal is in fact S , since it contains K , and so the right module $S/p(d)S$ is finite dimensional as a vector space. But $1 - ud$, and hence I , act as zero on any finite dimensional module. Thus $I \subset p(d)L$. Hence $B = p(d)S + I = p(d)S \cong S$ is projective.

Case 2 $I \not\subset B$. Since I is semisimple projective, $(B + I)/B \cong I/(B \cap I)$ is semisimple projective. But then $B + I \cong B \oplus (B + I)/B$. By Case 1, $B + I$ is projective, hence B is projective. This completes the proof that S is hereditary. \square

Definition 3.9. Let M_L be a module and $x \in L$. We write M^x for the set of x -torsion elements of M , i.e. $M^x = \{m \in M \mid mx^k = 0, k \gg 0\}$.

Lemma 3.10. Assume $r \neq 0$, let $L = L(-1, r, 0, \gamma)$ and let I be the two sided ideal generated by $1 - ud$. Let $J_m = \text{lann}_L(u^m)$. Then

- 1) $L^u = I$,
- 2) $J_m = L(1 - u^m d^m) = \bigoplus_{k=0}^{m-1} L(1 - ud)d^k$,
- 3) $\lim J_n = I$.

Proof. The first equality of 2) simply says that $1 - u^n d^n$ and $u^n d^n$ are idempotents. Half of the second equality is obvious and the other half follows from the equality $1 - u^m d^m = \sum_{k=0}^{m-1} u^k (1 - ud)d^k$. Directness of the sum follows from induction on m , via the observation

$$L(1 - u^{m-1} d^{m-1}) \cap L(1 - ud)d^{m-1} = 0.$$

This last equality is seen as follows: if $x(1 - u^{m-1} d^{m-1}) = y(1 - ud)d^{m-1}$, right multiply by u^{m-1} to get $0 = y(1 - ud)$, which is all that is required. The other two parts of the lemma follow immediately, since $\lim J_m = L^u$, but the limit of the right hand side of 2) is clearly I . \square

Remark 3.11. A similar statement holds for right ideals with d in place of u . It is also interesting to note that while every left ideal J_n is cyclic, ${}_L I$ is not a Noetherian module.

Proposition 3.12. *For $r \neq 0$, $L(c, r, 0, \gamma)$ has global dimension 2.*

Proof. We may assume $L = L(-1, r, 0, \gamma)$, so that our d, u -relation is $du - 1 = 0$. Let S be the subalgebra generated by d and u , as in lemma 3.8. Define $\theta \in \text{Aut}(S)$ by $\theta(d) = rd$ and $\theta(u) = r^{-1}u$. Define a θ -derivation δ by $\delta(d) = -\gamma d$ and $\delta(u) = r^{-1}\gamma u$. Then L is isomorphic to the Ore extension $S[h; \theta, \delta]$. Since S has global dimension 1, it follows from [16] Theorem 7.5.3 that $\text{gldim}L \leq 2$.

It now suffices to show that L has a proper right ideal that is not projective as a module. Our candidate is the right ideal $A := I + hL$, where I is the two sided ideal generated by $1 - ud$. Since $r \neq 0$, I is completely prime and so the kernel of the addition map $I \oplus hL \rightarrow A$, is isomorphic to $I \cap hL = hI$. Thus we have a short exact sequence

$$0 \rightarrow hI \rightarrow I \oplus hL \rightarrow A \rightarrow 0,$$

which we will prove does not split. Suppose that we have a splitting $f = f_1 \oplus f_2 : A \rightarrow I \oplus hL$. Then $f(h) = (0, h) + (x, -x)$ for some $x \in hI$. By 3.10, there exists m so that $xu^m = 0$. Then $f(hu^m) = (0, hu^m)$. In particular, $f_1(hu^mw) = 0$ and $f_2(hu^mw) = hu^mw$ for all $w \in L$. Consider just $f_2 : A \rightarrow hL$. Define $z = 1 - u^{m+1}d^{m+1}$ and notice that $zh = hz$ and $zu^m = u^m(1 - ud)$. Since $z \in A$, we may choose $y \in L$ with $f_2(z) = hy$. Then we have $hyhu^m = f_2(z)hu^m = f_2(zhu^m) = f_2(hu^m(1 - ud)) = hu^m(1 - ud) = zhu^m$, i.e. $(hy - z)hu^m = 0$. But then $(hy - z)u^m d^m h = 0$, and since h is regular and d is left regular, we get $(hy - z)u^m = 0$, i.e. $hy - z \in J_m$. We conclude that $z \in J_m + hL$. (Note: $J_m + hL$ is not an ideal, simply a vector space.) Since $J_m u^m = 0$, we now have $u^m(1 - ud) = zu^m \in hL$. But this is impossible, since $r \neq 0$, as it contradicts theorem 2.1. Hence the splitting f cannot exist. \square

Remark 3.13. Although it is not relevant to the proof above, it is worth noting that I_L is projective and thus, since h is regular, the short exact sequence given in the proof is a projective resolution of $I + hL$.

It remains only to consider the cases where where $r = s = 0$ and $f = c$. These algebras are all isomorphic and all have global dimension 1. Our final theorem and proof of the section are closely related to 3.8.

Theorem 3.14. *The algebras $L(c, 0, 0, \gamma)$ are hereditary.*

Proof. Without loss of generality we may assume $\gamma = 0$ and $c = -1$ and take $L = L(-1, 0, 0, 0)$. Let I be the two sided ideal generated by the idempotent $1 - ud$ and let D be the right L -module hL . Notice that $h = (1 - ud)h$ is in I , so that $L/I \cong K[d, d^{-1}]$ is a P.I.D. Since $1 - ud$ and ud are orthogonal idempotents and $du = 1$, we have $\text{rann}_L(1 - ud) = uL$. By 2.1, $\text{rann}_L(h) = uL$. Similarly, $\text{rann}_L(u^k(1 - ud)) = \text{rann}_L(u^k h) = uL$. Thus the right ideals $u^j(1 - ud)L$ and $u^j hL$, $j \geq 0$, are all isomorphic, as right L -modules, to D . In particular D is projective, since $(1 - ud)L$ is projective. Notice that D has the K -basis $\{h^k d^m \mid m \geq 0, k \geq 1\}$.

Now let $J < D$ be any nonzero right submodule. We claim that $J \cong D$ as modules. It is clear that $J \cap K[h]$ is a $K[h]$ -ideal inside $hK[h]$. Let $0 \neq x \in J$. Then $x = \sum_0^m p_k(h)d^k$ where each $p_k(h) \in K[h]h$. Let y be a $K[h]$ -generator for the $K[h]$ ideal $J \cap K[h]$. We will show $J = yL$. Since $0 \neq xu^m = p_m(h) \in J \cap K[h]$, so $y \neq 0$ and $p_m(h) = yp'_m(h)$. Then $x - p_m(h)d^m = x - yp'_m(h)d^m \in J$ and by induction on m , $x - p_m(h)d^m \in yL$. Thus $x \in yL$ and $yL = J$. Finally, $y \in K[h]h$, from which it is clear that $\text{rann}(y) = uL$. Thus $J \cong D$.

Now, let B be any right ideal in S and consider three cases. Case 1 $B \subset I$. Let $I_m = \bigoplus_{k=0}^{m-1} u^k(1 - ud)L \subset I$. Then I_m is a filtration of I and $I_m/I_{m-1} \cong u^m(1 - ud) \cong D$ for all m . Let $B_m = B \cap I_m$, a filtration of B . For all m , let $\bar{B}_m = B_m/B_{m-1}$. Then \bar{B}_m is isomorphic to a submodule of D and is thus either 0 or isomorphic to D , in particular it is projective. It follows that $B \cong \bigoplus_m \bar{B}_m$ and B is projective.

Case 2 $I \subset B$. We may assume $B \neq I$. Then B/I is a nonzero right ideal in L/I . Since L/I is a P.I.D. we may write B as $p(d)L + I$ for some $p \in K[d]$ with $p(0) \neq 0$. Just as in the proof of 3.8, we have $p(d)L \cong L$.

Now we show that $I \subset p(d)L$. Let p have degree m and notice that since $dh = 0$, $p(d)h = p(0)h \neq 0$. From this and the argument of Case 1 of 3.8, it follows that the vector space $p(d)L + K + Kd + Kd^2 + \dots + Kd^{m-1}$ forms a right ideal in L . But this ideal must be L , since it contains K , and so the right module $L/p(d)L$ is finite dimensional as a vectorspace. But I acts as zero on any finite dimensional module, so $I \subset p(d)L$. Hence $B = p(d)L + I = p(d)L \cong L$ is projective.

Case 3 B is arbitrary. Then $B+I$ and $B \cap I$ are projective and hence $B \oplus I$ is projective, as required. \square

4. WEIGHT MODULES, SIMPLE WEIGHT MODULES AND FINITE DIMENSIONAL SIMPLE MODULES.

Throughout this section we take $L = L(f, r, s, \gamma)$ and we assume $rs \neq 0$, so that L is a Noetherian domain. We begin by discussing briefly the \mathbb{Z} -grading on L defined by giving the generators u, h, d the degrees 1, 0, -1 respectively. This grading is will be denoted $L = \bigoplus_{k \in \mathbb{Z}} L_k$.

Proposition 4.1. *The subalgebra L_0 has K -basis $X = \{(ud)^i h^j \mid i, j \geq 0\}$. It is generated as an algebra by h and ud and is a polynomial ring on those two variables. If $m \geq 0$, $L_m = u^m L_0 = L_0 u^m$. If $m < 0$, $L_m = d^{-m} L_0 = L_0 d^{-m}$.*

Proof. The basis statement is Remark 2.2 and the fact that L_0 is generated by h and ud follows. The basis then shows L_0 has GK-dimension 2, and so must be a polynomial ring. From 2.2 it is now clear that $L_m = u^m L_0$ for $m \geq 0$. One sees that $u^m L_0 = L_0 u^m$ by a simple computation. Similarly, for $m < 0$, $L_m = d^{-m} L_0 = L_0 d^{-m}$. \square

Remark 4.2. It is easy to see that when r is not a root of unity, L_0 is exactly the commutator of h in L . On the other hand, if r is a primitive m^{th} root of unity, then the commutator of h in L is the m^{th} Veronese subalgebra of L .

We denote by \widehat{L}_0 the set of one-dimensional characters of L_0 , which we may identify with the dual of the linear span of h and ud . For $\zeta \in \widehat{L}_0$, we will write $\zeta = (\alpha, \beta)$ to denote $\zeta(h) = \alpha$ and $\zeta(ud) = \beta$.

Let $\Phi : \widehat{L}_0 \rightarrow \widehat{L}_0$ be the non-linear discrete dynamical system given by $\Phi(\alpha, \beta) = (r\alpha - \gamma, s\beta - f(\alpha))$. We note that Φ has a polynomial inverse $\Phi^{-1}(\alpha, \beta) = (\frac{\alpha+\gamma}{r}, \frac{\beta+f(\frac{\alpha+\gamma}{r})}{s})$. We denote the (full) orbit of ζ under Φ by $\langle \zeta \rangle = \{\Phi^k(\zeta) | k \in \mathbb{Z}\}$. Whenever we have a fixed $\zeta = (\alpha, \beta)$ we write $\zeta_n = (\alpha_n, \beta_n)$ for $\Phi^n(\zeta)$, $n \in \mathbb{Z}$.

For any left L -module M and $\zeta \in \widehat{L}_0$, let $M_\zeta = \{m \in M | xm = \zeta(x)m, \forall x \in L_0\}$ be the ζ -weight space of M . We say that M is a weight module if $M = \sum_{\zeta \in \widehat{L}_0} M_\zeta$. The set of $\zeta \in \widehat{L}_0$ for which $M_\zeta \neq 0$ is called the set of weights of M and is denoted $wt(M)$.

We can now construct the *Universal weight module* with weight $\zeta \in \widehat{L}_0$, $W(\zeta) := L \otimes_{L_0} K_\zeta$, where K_ζ is the one-dimensional L_0 module given by the character ζ . We fix a basis vector v_ζ for K_ζ and put $e_0 = e_0(\zeta) = 1 \otimes v_\zeta \in W(\zeta)$. For $m > 0$, put $e_m = u^m e_0$ and $e_{-m} = d^m e_0$. Then by 2.2, $\{e_m | m \in \mathbb{Z}\}$ is a basis for $W(\zeta)$.

Lemma 4.3. *The module $W(\zeta)$ is a weight module with set of weights $wt(W(\zeta)) = \langle \zeta \rangle$. In particular, $e_m \in W(\zeta)_{\zeta_m}$ for $m \in \mathbb{Z}$. If the orbit $\langle \zeta \rangle$ is infinite, then all of the weight spaces of $W(\zeta)$ are one dimensional.*

Proof. It suffices to prove $e_m \in W(\zeta)_{\zeta_m}$. This is true by definition for $m = 0$. For $m > 0$ we have, by induction, $he_m = hue_{m-1} = (ruh - \gamma u)e_{m-1} = u(rh - \gamma)e_{m-1} = (r\alpha_{m-1} - \gamma)ue_{m-1} = \alpha_m e_m$. Similarly $ude_m = udue_{m-1} = u(sud - f(h))e_{m-1} = u(s\beta_{m-1} - f(\alpha_{m-1}))e_{m-1} = \beta_m e_m$, as required. The appropriate formulas for $m < 0$ are proved by similar computations. \square

Remark 4.4. We note that du is in L_0 and observe the following formula for later use: $(du)e_m = \beta_{m+1}e_m$ for all $m \in \mathbb{Z}$. We also record for later use the action of d and u on these elements. For $m > 0$, $ue_m = e_{m+1}$ and $de_m = due_{m-1} = \beta_m e_{m-1}$. For $m < 0$, $de_m = e_{m-1}$ and $ue_m = ude_{m+1} = \beta_{m+1}e_{m+1}$.

We need one more piece of unfortunate notation. For fixed $\zeta \in \widehat{L}_0$, let $B_+(\zeta)$ be $\{\min\{k > 0 | \beta_k = 0\}\}$ if the minimum exists. Otherwise $B_+(\zeta) = \emptyset$. Similarly $B_-(\zeta) = \{\max\{k \leq 0 | \beta_k = 0\}\}$ if the maximum exists. Otherwise $B_-(\zeta) = \emptyset$. Put $B(\zeta) = B_+(\zeta) \cup B_-(\zeta)$.

We can now classify all simple weight modules, up to a basic understanding of the dynamical system Φ . This includes a classification of all finite dimensional simple modules. The reader will immediately observe the emergence of classical highest and lowest weight modules in the theorems that follow. However, we begin by analyzing a collection of finite dimensional modules that are not highest or lowest weight modules.

Lemma 4.5. *Assume the order of the orbit $\langle \zeta \rangle$ is p . Then the actions of u^p and d^p on $W(\zeta)$ commute with the action of L . Moreover, for any non-zero scalar λ , the submodules $L(u^p - \lambda)e_0 = (u^p - \lambda)W(\zeta)$ and $L(d^p - \lambda)e_0 = (d^p - \lambda)W(\zeta)$ have codimension p .*

Proof. Define scalars μ_k and τ_k by $u^p e_k = \mu_k e_{k+p}$ and $u^{p-1} e_k = \tau_k e_{k+p-1}$. Note that μ_k or τ_k may be 0 for some k . Also, $\alpha_{k+p} = \alpha_k$ and $\beta_{k+p} = \beta_k$ for all k . Thus, $uu^p e_k = u^p u e_k$, $hu^p e_k = h\mu_k e_{k+p} = \alpha_{k+p} \mu_k e_{k+p} = \alpha_k u^p e_k$ and $du^p e_k = (du)\tau_k e_{k+p-1} = \beta_{k+p} \tau_k e_{k+p-1} = u^{p-1} \beta_k e_k = u^{p-1}(ud)e_k = u^p d e_k$. This shows that the action of u^p commutes with L . It follows for any $\lambda \in K$ that $L(u^p - \lambda)e_0 = (u^p - \lambda)W(\zeta)$. A similar statement holds for d^p .

Let $\Omega = \prod_0^{p-1} \beta_i$ and let $\lambda \in K^\times$. Applying $(u^p - \lambda)$ to the vectors e_k yields the following spanning set for $(u^p - \lambda)W(\zeta)$: $\{e_{m+p} - \lambda e_m | m \geq 0\} \cup \{\Omega e_{m+p} - \lambda e_m | m \leq -p\} \cup \{\mu_{k-p} e_k - \lambda e_{k-p} | 1 \leq k \leq p-1\}$. Since $\lambda \neq 0$, these vectors are linearly independent, even if $\Omega = 0$. The codimension statement is now clear. The proof of the statement with d in place of u is a similar computation. \square

Definition 4.6. In the context of Lemma 4.5, that is $|\langle \zeta \rangle| = p$ and $\lambda \in K^\times$, we define p -dimensional L modules $F_c(\zeta, \lambda) = W(\zeta)/(u^p - \lambda)W(\zeta)$ and $\bar{F}_c(\zeta, \lambda) = W(\zeta)/(d^p - \lambda)W(\zeta)$. (The subscript c is intended to remind us that the orbit of ζ is cyclic.)

Lemma 4.7. *Assume $\langle \zeta \rangle$ has order p . Let $\Omega = \prod_0^{p-1} \beta_i$.*

- 1) $F_c(\zeta, \lambda)$ and $\bar{F}_c(\zeta, \lambda)$ are both simple modules of dimension p .
- 2) The modules $F_c(\zeta, \lambda)$ are mutually nonisomorphic as λ varies through K^\times .

3) The modules $\bar{F}_c(\zeta, \lambda)$ are mutually nonisomorphic as λ varies through K^\times .

4) If $\Omega \neq 0$, then $F_c(\zeta, \lambda) \cong \bar{F}_c(\zeta, \Omega/\lambda)$.

5) If $\Omega = 0$, then the modules $F_c(\zeta, \lambda)$ and $\bar{F}_c(\zeta, \lambda')$ are non-isomorphic for all $\lambda, \lambda' \in K^\times$.

Proof. The module $F_c(\zeta, \lambda)$ is L_0 -semisimple, has dimension p , and has p distinct weight spaces which are permuted cyclically by u . It follows immediately that the module is simple. Note that d may well annihilate one or more of the weight spaces, so d does not necessarily permute the weight spaces. However, d does cyclically permute the weight spaces of $\bar{F}_c(\zeta, \lambda)$, so that module is simple as well. This is 1).

The modules $F_c(\zeta, \lambda)$ and $\bar{F}_c(\zeta, \lambda)$ determine λ as the eigenvalue of u^p or d^p respectively. This is 2) and 3).

In general, we can compute the action of d^p on $F_c(\zeta, \lambda)$. Let v_i be the image of e_i in the module for $0 \leq i < p$. Then the v_i are a basis. We have $uv_i = v_{i+1}$ for $i < p-1$ and $uv_{p-1} = \lambda v_0$. Then $dv_i = \beta_i v_{i-1}$ for $i > 0$ and $dv_0 = (1/\lambda)du(u^{p-1})v_0 = (1/\lambda)duv_{p-1} = (\beta_p/\lambda)v_{p-1} = (\beta_0/\lambda)v_{p-1}$. Thus $d^p v_i = (\Omega/\lambda)v_i$ for all i . One computes similarly that u^p acts as (Ω/λ') on the module $\bar{F}_c(\zeta, \lambda')$. This immediately gives 4) and 5). \square

We now turn to objects akin to classical highest and lowest weight modules. Fix ζ and suppose that $\beta_j = 0$ for some $j > 0$. Then $de_j = \beta_j e_{j-1} = 0$, from which it follows easily that Le_j , the span of $\{e_k | k \geq j\}$, is a cyclic submodule of $W(\zeta)$. Since d annihilates the cyclic vector e_j , one would call this a *lowest weight module* in the classical language. We also obtain a new module $W(\zeta)/Le_j$. One expects the vector v_{j-1} , the image of e_{j-1} in this module to be a *highest* weight vector, but this may not be the case, as v_{j-1} may not be a cyclic vector for the module. There is one case, however, when $W(\zeta)/Le_j$ is a highest weight module, when $B_+(\zeta) = \{j\}$, i.e. when j is the smallest positive integer for which $\beta_j = 0$. This analysis, and its dual for $B_-(\zeta)$ prompt the following definitions.

Definition 4.8. Fix $\zeta \in \widehat{L}_0$.

- 1) If $B_+(\zeta) = \{j\}$ is nonempty, we put $M_+(\zeta) = W(\zeta)/Le_j$.
- 2) If $B_-(\zeta) = \{i\}$ is nonempty, we put $M_-(\zeta) = W(\zeta)/Le_{i-1}$.
- 3) If $B(\zeta) = \{i, j\}$ has two elements, we put $F_{hw}(\zeta) = W(\zeta)/(Le_j + Le_{i-1})$. (The subscript hw is to remind us that this finite dimensional module is a classical highest and lowest weight module.)

Lemma 4.9. *Fix $\zeta \in \widehat{L}_0$.*

- 1) *If $B(\zeta) = B_+(\zeta) = \{j\}$, then the orbit $\langle \zeta \rangle$ is infinite and the module $M_+(\zeta)$ is simple and infinite dimensional.*
- 2) *If $B(\zeta) = B_-(\zeta) = \{i\}$, then the orbit $\langle \zeta \rangle$ is infinite and the module $M_-(\zeta)$ is simple and infinite dimensional.*
- 3) *If $B(\zeta) = \{j, i \mid i < j\}$ has two elements, then the module $F_{hw}(\zeta)$ is simple of dimension $j - i$.*

Proof. If $B(\zeta)$ has one element, then the orbit $\langle \zeta \rangle$ must be infinite. Hence the weight spaces of either $M_+(\zeta)$ or $M_-(\zeta)$ are all one-dimensional. To see that the module is simple it suffices to see that any non-zero weight vector is a cyclic vector. In case 1) let v_k be the image of e_k in $M_+(\zeta)$, for all $k < j$. These vectors form a basis of $M_+(\zeta)$. Since $B(\zeta) = \{j\}$, uv_k is a nonzero multiple of v_{k+1} for all $k < j - 1$, and thus $u^{-k+j-1}v_k$ is a nonzero multiple of v_{j-1} . But $B(\zeta) = \{j\}$ also tells us that d acts injectively on the module, and thus v_{j-1} is a cyclic vector. This proves 1) and a similar analysis proves both 2) and 3). We note that $uv_{j-1} = 0$, so v_{j-1} is a true *highest* weight vector in $M_+(\zeta)$. \square

We can now give a complete classification of simple weight modules.

Theorem 4.10. *Let N be a simple weight module for L . Fix a weight $\zeta \in wt(N)$. Then exactly one of the following occurs:*

- 1) *$\langle \zeta \rangle$ is infinite and $B(\zeta) = \emptyset$, in which case $N \cong W(\zeta)$.*
- 2) *$B(\zeta) = B_+(\zeta) = \{j\}$, in which case $N \cong M_+(\zeta)$ and N is infinite dimensional.*
- 3) *$B(\zeta) = B_-(\zeta) = \{i\}$, in which case $N \cong M_-(\zeta)$ and N is infinite dimensional.*
- 4) *$\langle \zeta \rangle$ is infinite and $B(\zeta) = \{i, j\}$ ($i < j$), in which case $N \cong F_{hw}(\zeta)$ and $\dim(N) = j - i$.*

5) $\langle \zeta \rangle$ is finite of order p and $B(\zeta) = \emptyset$, in which case there exists unique $\lambda \in K^\times$ for which $N \cong F_c(\zeta, \lambda)$ and $\dim(N) = p$.

6) $\langle \zeta \rangle$ is finite of order p and $B(\zeta) = \{i, j\}$, in which case N is isomorphic to exactly one of the following simple modules:

- (i) $F_c(\zeta, \lambda)$ for some $\lambda \in K^\times$,
- (ii) $\bar{F}_c(\zeta, \mu)$ for some $\mu \in K^\times$,
- (iii) $F_{hw}(\zeta)$.

Proof. Since N is simple and $N_\zeta \neq 0$, we have a surjective homomorphism $G : W(\zeta) \rightarrow N$. For all $m \in \mathbb{Z}$, let $v_m \in N$ be the image of e_m . Then the v_m span N . Recalling from 4.4 the action of d and u on e_m , we get: $uv_m = v_{m+1}$ and $dv_m = duv_{m-1} = \beta_m v_{m-1}$ for $m > 0$. For $m < 0$, $dv_m = v_{m-1}$ and $uv_m = udv_{m+1} = \beta_{m+1} v_{m+1}$.

Since it is not possible to have $\langle \zeta \rangle$ finite and $B(\zeta)$ of order one at the same time, it is clear that the 6 possibilities for the combinatorial pair $(\langle \zeta \rangle, B(\zeta))$ are exhaustive and mutually exclusive.

Suppose $B(\zeta) = \emptyset$. Then d and u act injectively on $W(\zeta)$ and every e_i is a cyclic vector for $W(\zeta)$. If we additionally assume that $\langle \zeta \rangle$ is infinite, then the weight spaces of $W(\zeta)$ are all one dimensional and d and u , together, transitively permute these weight spaces. This forces $W(\zeta)$ to be simple and yields 1). So we may assume instead that $\langle \zeta \rangle$ has order p . By 4.5, $W(\zeta)$ is not simple and so $M = \ker(G)$ is not zero. Amongst all non-zero elements of M , choose one, $w = \sum_i a_i e_i$ which has the least possible number of non-zero coefficients. Since each e_i is a cyclic vector, this w cannot be just $a_i e_i$ for any i . By the minimality of w , all the e_i occurring nontrivially in w must have the same weight. Multiplying by d^k or u^k as necessary, we may assume $w = \sum_{i=0}^{i=m} a_i e_{ip}$ with $a_0 \neq 0$. Let $q(u^p) = \sum_{i=0}^{i=m} a_i u^{ip}$. We have $L \cdot q(u^p)e_0 \subset M$. But by 4.5, $L \cdot q(u^p)e_0 = q(u^p)W(\zeta)$. Thus $q(u^p)N = 0$. Factor $q(u^p)$ as $b \prod_k (u^p - \lambda_k)$, where, since $a_0 \neq 0$, each $\lambda_k \neq 0$. By 4.5 again, each $(u^p - \lambda_k)N$ is a submodule of N and is thus N or 0 . Clearly then for some (necessarily unique) λ , $(u^p - \lambda)N = 0$. But now we can factor G through $F_c(\zeta, \lambda)$, which is simple, and we have obtained 5).

Suppose next that $B(\zeta) = B_+(\zeta) = \{j\}$. In particular then $\langle \zeta \rangle$ is an infinite orbit. We claim that $v_m = 0$ for $m \geq j$. To see this,

assume it is not true and let A be the linear span of $\{v_m | m \geq j\}$. Since $dv_j = \beta_j v_{j-1} = 0$, A is a nonzero submodule and must therefore be N . Then v_0 is in A and so the weight of v_0 is the same as the weight of some v_k , $k \geq j$. This contradicts the infinite orbit condition and proves the claim. But now the map G factors through the simple module $M_+(\zeta)$, proving 2).

The proofs of 3) and 4) are essentially the same as 2).

Finally, we may assume $\langle \zeta \rangle$ is finite of order p and $B(\zeta) = \{i, j\}$. Note that $p \geq j$ and $-p \leq i$. Let A_+ be the span of the vectors $\{v_k | k \geq j\}$ and let A_- be the span of the vectors $\{v_k | k < i\}$. Then A_+ and A_- are submodules of N and each is either 0 or N .

Suppose A_- and A_+ are both 0. Then G factors through $F_{hw}(\zeta)$ and we have established 6(iii).

Suppose $A_- = 0$ and $A_+ = N$. Then $v_0 \in A_+$, so we have $v_0 + \sum_{i=1}^m a_i v_{ip} = 0$ for some scalars a_i . Put $q(u^p) := 1 - \sum a_i u^{ip}$ and factor this as $b \prod_k (u^p - \lambda_k)$. As we have seen previously, $q(u^p)N = 0$, and thus $(u^p - \lambda_k)N = 0$ for one of the λ_k . Hence G factors through $F_c(\zeta, \lambda_k)$ and we have established 6(i). Similarly, if $A_- = N$ and $A_+ = 0$ then G factors through $\bar{F}_c(\zeta, \mu_k)$ and we have 6(ii).

The only possibility left is $A_- = A_+ = N$. The previous paragraph would then establish that N is isomorphic to both $F_c(\zeta, \lambda)$ and $\bar{F}_c(\zeta, \mu)$ for some scalars λ and μ . From 4.7 we know this is impossible and hence the last possibility does not occur. \square

Remark 4.11. Cases 4, 5 and 6 of the previous theorem provide a classification of finite dimensional simple L -modules.

Corollary 4.12. *The simple modules $F_c(\zeta, \lambda)$ and $\bar{F}_c(\zeta, \mu)$ depend on the orbit $\langle \zeta \rangle$ but not on the particular ζ in that orbit.*

We conclude this section with the appropriate simple combinatorics required to express the classification theorem in terms of orbits $\langle \zeta \rangle$ rather than individual weights. To do so, we need the following definition which explains how an orbit $\langle \zeta \rangle$ should be partitioned to correspond to simple modules.

Definition 4.13. Fix an orbit $\langle \zeta \rangle$ and choose some $\zeta_0 \in \langle \zeta \rangle$. A set of consecutive indices $I \subset \mathbb{Z}$ is a β -set for $\langle \zeta \rangle$ with respect to ζ_0 if I is a maximal subset of \mathbb{Z} with respect to the property: $k, l \in I$ and $k < l$ implies $\beta_l \neq 0$. To each β -set I we put $\langle \zeta \rangle_I := \{\zeta_k | k \in I\}$ and refer to this set as a β -block for the orbit $\langle \zeta \rangle$.

It is clear that the β -sets depend on the choice of ζ_0 in the orbit $\langle \zeta \rangle$, but the β -blocks do not depend on ζ_0 . Further, the orbit $\langle \zeta \rangle$ is the disjoint union of all of its β -blocks.

Corollary 4.14. *Fix an orbit $\langle \zeta \rangle$.*

- (1) *If $\langle \zeta \rangle$ is infinite and has only one β -set, $I = \mathbb{Z}$, then $\langle \zeta \rangle_I = \langle \zeta \rangle$ is the set of weights of the simple module $W(\zeta)$. In particular $W(\zeta) \cong W(\mu)$ for all $\mu \in \langle \zeta \rangle$.*
- (2) *If $\langle \zeta \rangle$ is infinite and I is a β -set properly contained in \mathbb{Z} , then each $\langle \zeta \rangle_I$ is the set of weights of one of the simple modules $F_{hw}(\mu)$, $M_+(\mu)$ or $M_-(\mu)$ for any $\mu \in \langle \zeta \rangle_I$. The dimension of the module is the order of the β -block $\langle \zeta \rangle_I$.*
- (3) *If $\langle \zeta \rangle$ is of finite order and has only one β -set, $I = \mathbb{Z}$, then $\langle \zeta \rangle_I = \langle \zeta \rangle$ is the set of weights of any of the simple modules $F_c(\zeta, \lambda)$, $\lambda \in K^\times$.*
- (4) *If $\langle \zeta \rangle$ is of finite order and has more than one β -set I , then the full orbit $\langle \zeta \rangle$ is the set of weights of each of the (non-isomorphic) simple modules $F_c(\zeta, \lambda)$ and $\bar{F}_c(\zeta, \lambda)$ for every $\lambda \in K^\times$. In addition, each β -block $\langle \zeta \rangle_I$ is the set of weights of $F_{hw}(\mu)$ for any $\mu \in \langle \zeta \rangle_I$.*

We note in part 4 of the corollary that it is possible for the finite orbit $\langle \zeta \rangle$ to have only one β -block.

Remark 4.15. The result above is very similar to a result of Bavula, [2]. However, Bavula works with a generalized Weyl algebra $D(\sigma, a)$ whose ground ring D is Dedekind and with the hypothesis that the dynamical system on $\text{Spec}(D)$, induced from $\sigma \in \text{Aut}(D)$, has no finite orbits. In our setting, the ground ring is the polynomial ring $K[ud, h]$ and the induced dynamical system Φ may well have finite

orbits. Nevertheless, our result is just a generalization of Bavula's. It is easy to formulate and prove a very general version of Bavula's result for any commutative K -algebra and automorphism σ .

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