

GLOBAL DIMENSION FOUR EXTENSIONS OF ARTIN-SCHELTER REGULAR ALGEBRAS

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ABSTRACT. This paper classifies central and normal extensions from global dimension three Artin-Schelter regular algebras to global dimension four Artin-Schelter regular algebras. Let A be an AS regular algebra of global dimension three, and let D be an extension of A by a normal graded element z , i.e. $D/\langle z \rangle = A$. The algebra A falls under a classification due to Artin, Schelter, Tate and Van den Bergh [1, 2, 3], and is either quadratic or cubic. The quadratic algebras A are Koszul, and this fact was used by Le Bruyn, Smith and Van den Bergh in [5] to classify the 4-dimensional AS regular algebras D when A is quadratic and $\deg(z) = 1$. Alternative methods are needed when A is cubic or $\deg(z) > 1$. We prove in all such cases that the regularity of D and z is equivalent to the regularity of z in low degree (e.g. 2 or 3) and this is equivalent to easily verifiable matrix conditions on the relations for D .

1. INTRODUCTION

The concept of an Artin-Schelter regular (AS regular), graded k -algebra was defined by M. Artin and W. Schelter in [1]. The connected AS regular algebras of global dimension three which are generated by degree one elements were classified into quadratic and cubic families in [1, 2, 3]. The AS regular algebras of global dimension four have proven to be more difficult to classify. One can begin by looking for the 4-dimensional regular algebras which are extensions of 3-dimensional regular algebras. We call an algebra D with graded central element z such that $A \cong D/\langle z \rangle$ a *central extension* of A . One notices that for any AS regular algebra A of global dimension 3, the central extension $D := A \otimes_k k[z]$ is an AS regular algebra of global

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dimension 4. This observation leads to the question: given a central extension D of a 3-dimensional AS regular algebra, is D itself AS regular? Moreover, one can generalize and ask the same question of *normal extensions*, that is, algebras D with a normal graded element z such that $A \cong D/\langle z \rangle$. In particular, can one determine regularity from the defining relations for D ? This question was partially answered by Le Bruyn, Smith and Van den Bergh in [5], where central extensions of quadratic algebras by elements of degree one were studied. Le Bruyn et. al. show that D is AS regular and z is a regular element of D if and only if there exists a solution to a certain system of linear equations. We improve this result by showing that the regularity of D and of z are equivalent to the regularity of z on the degree 2 part of D .

Theorem 1.1. *Let $z \in D_1$ be central and let $A = D/\langle z \rangle$ be a quadratic AS regular algebra of global dimension 3 defined by relations $F = MX$, where $X = (x_1, x_2, x_3)^t$ and $(X^t M)^t = QMX$ for Q in $Gl_3(k)$ (cf. [1]). Write $D = k\langle x_1, x_2, x_3, z \rangle / \langle r_1, r_2, r_3, zx_i - x_i z \rangle$ where $(r_1, r_2, r_3)^t = F + Ez$ for E in $k\langle x_1, x_2, x_3, z \rangle_1^3$. The following are equivalent:*

- (1) D is an Artin-Schelter regular algebra (of global dimension 4).
- (2) z is a regular element of D .
- (3) z is a 2-regular element of D .
- (4) $E^t Q^t X = X^t E$ in D .

The quadratic family of global dimension 3 AS regular algebras are Koszul algebras, and this fact was exploited in [5]. Alternative techniques are needed to study the cubic family, since these algebras are not Koszul. Nonetheless, a parallel theorem holds for the cubic family.

Theorem 1.2. *Let $z \in D_1$ be central and let $A = D/\langle z \rangle$ be a cubic AS regular algebra of global dimension 3 defined by relations $F = MX$, where $X = (x_1, x_2)^t$ and $(X^t M)^t = QMX$ for Q in $Gl_2(k)$. Write $D = k\langle x_1, x_2, z \rangle / \langle r_1, r_2, zx_i - x_i z \rangle$ where $(r_1, r_2)^t = F + Ez$ for E in $k\langle x_1, x_2, z \rangle_2^2$. The following are equivalent:*

- (1) D is an Artin-Schelter regular algebra (of global dimension 4).
- (2) z is a regular element of D .
- (3) z is a 3-regular element of D .
- (4) $E^t Q^t X = X^t E$ in D .

In [7] and [8] Stephenson extended the classification from [1, 2, 3] and showed there are AS regular algebras of global dimension three generated in degrees other than one. It is therefore natural to ask whether theorems similar to 1.1 and 1.2 hold for central extension by elements of degree greater than one. We show that such theorems hold for z of any degree. For example, we prove:

Theorem 1.3. *Let $z \in D_2$ be central, $A = D/\langle z \rangle$ be a quadratic AS regular algebra of global dimension 3 defined by relations $F = MX$ where $X = (x_1, x_2, x_3)^t$ and $(X^t M)^t = QMX$ for Q in $Gl_3(k)$. Write $D = k\langle x_1, x_2, x_3, z \rangle / \langle r_1, r_2, r_3, x_i z - z x_i \rangle$ where $(r_1, r_2, r_3)^t = F + Ez$ for E in k^3 . The following are equivalent:*

- (1) D is an Artin-Schelter regular algebra (of global dimension 4).
- (2) z is a regular element of D .
- (3) z is a 1-regular element of D .
- (4) $QE = E$.

It is interesting to observe that unless $E = 0$, the algebras classified by Theorem 1.3 are in fact generated by degree one elements, even though the extending element has degree two.

Since normal extensions are a generalization of central extensions, we look for corresponding theorems about normal extensions. Any AS regular algebra, which is a normal extension by an element of degree one, is a twist of a central extension. One might wonder if this situation would be repeated in higher degrees, that is, if normal extensions would be twists of central extensions. In Section 5 we show that if so, the twisting system is by no means obvious. In example 5.1 we exhibit an algebra D which is a normal extension by a degree two element z . We show that D is not a twist by an automorphism of an extension in which z is central. However, D is AS regular, which we demonstrate using the following theorem.

Theorem 1.4. *Let $A = D/\langle z \rangle$ be a cubic AS regular algebra of global dimension 3 defined by relations $F = MX$, where $(X^t M)^t = QMX$ for Q in $Gl_2(k)$. Let $z \in D_2$ be normal and define a matrix N in $GL_2(k)$ by $Xz = zNX$ where $X = (x_1, x_2)^t$. Let $\tilde{T} = k\langle x_1, x_2, z \rangle / \langle n_1, n_2 \rangle$ where $(n_1, n_2) = (Xz - zNX)^t$. Write $D = \tilde{T} / \langle r_1, r_2 \rangle$ where $(r_1, r_2)^t = F + \tilde{E}Xz$ for \tilde{E} in $M_2(k)$. The following are equivalent:*

- (1) D is an Artin-Schelter regular algebra (of global dimension 4).
- (2) z is a regular element of D .
- (3) z is a 3-regular element of D .
- (4) There exists a matrix G in $Gl_2(k)$ such that
 - (i) $zF = GFz$ in \tilde{T}^2 ,
 - (ii) $G\tilde{E}N = \tilde{E}$ in $M_2(k)$,
 - (iii) $\tilde{N}^t E^t = Q\tilde{E}$ in $M_2(k)$.

In fact, analogous four-part theorems hold in both the quadratic and cubic families of algebras for a normal $z \in D$ of arbitrary degree. Only the minimal degree of regularity for z and the explicit matrix conditions change. These individual theorems are stated in the subsections of sections 3 and 4. It is worth noting that, given the relations for an algebra D , the fourth condition of our theorems can be easily checked and examples can be quickly generated.

Section 2 contains some technical definitions and the notation which will be used throughout the paper. In section 3 we study extensions of the cubic AS regular algebras, beginning with extensions by elements of degree greater than three. In section 4 we identify the AS regular extensions of the quadratic family, again beginning with z of the highest degree and working down. Section 5 contains the non-twist example mentioned above, as well as other examples of AS regular algebras which can be found using these theorems.

2. DEFINITIONS AND NOTATION

Throughout the paper, k denotes an algebraically closed field of characteristic different from two. All algebras herein are assumed to be finitely generated, connected, \mathbb{N} -graded k -algebras. Grading will be denoted by subscripts.

Definition 2.1. [1, Page 171] A connected \mathbb{N} -graded k -algebra B is called Artin-Schelter Regular (AS regular) if

- (a) the global (homological) dimension of B ($\text{gldim}(B)$) is finite,
- (b) the Gelfand-Kirillov dimension of B ($\text{GKdim}(B)$) is finite, and
- (c) B is Gorenstein; that is, $\text{Ext}_B^n(k, B) = \delta_n^q k$ where $n = \text{gldim}(B)$.

The Gorenstein condition can be understood to say that if \mathbf{P}^\bullet is a projective resolution of the left B -module k , then the dual complex $\text{Hom}_B(\mathbf{P}^\bullet, B)$ is a projective resolution of a shift of the right B -module k .

Definition 2.2. A homogeneous element z of a graded algebra D is said to be regular if it is neither a left nor a right zero divisor. We say z is n -regular if both left and right multiplication by z is injective on D_n .

Throughout this paper A will be an AS regular algebra of global dimension 3 with fixed generators in A_1 , and D will be a k -algebra with graded normal element z such that $D/\langle z \rangle = A$. From Levasseur [4] we know that the algebras A are Noetherian domains.

Remark 2.3. Since A is a domain which is generated by degree one elements, and z is a normal element of D , it follows that if z is n -regular then z is m -regular for all $m \leq n$.

Since z is normal and $D/\langle z \rangle = A$, we define the functor $\pi : \{\text{graded left } D\text{-modules}\} \rightarrow \{\text{graded left } A\text{-modules}\}$ by $M \mapsto M/Mz$.

If A is cubic, we will write the generators as $\{x_1, x_2\}$; when A is quadratic the generators will be $\{x_1, x_2, x_3\}$. Let $j \in \{2, 3\}$ be the number of generators of A , and let $X = (x_1, \dots, x_j)^t$. According to [1] we can choose a basis for the vector space of relations, say $\{f_i\}$, so that if $F = (f_1, \dots, f_j)^t$, there is a $j \times j$ matrix M with homogeneous entries, and a matrix Q in $GL_j(k)$ such that $F = MX$ and $X^t M = (QF)^t$.

Let T be the free algebra on the generators of A and on an additional element z of degree α , and let \tilde{T} be a quotient of T wherein the image of z is normal, that is $\tilde{T} = T/\langle n_1, \dots, n_j \rangle$ where $(n_1, \dots, n_j)^t = Xz - zNX$ for some N in $M_j(k)$. Notice that if N is not invertible, then z is not 1-regular. It will be shown, without reference to N , that a necessary condition for D to be AS regular is that z be regular, and therefore we restrict our attention to the cases where $N \in GL_j(k)$. For convenience we will write N^{-t} to denote $(N^t)^{-1}$. The action of $GL_j(k)$ on $k\langle x_1, \dots, x_j \rangle_1$ induces an action on $k\langle x_1, \dots, x_j \rangle$ which we will denote by $u, N \mapsto u \circ N$ for $u \in k\langle x_1, \dots, x_j \rangle$. Thus $M \circ N$ denotes the matrix whose (i, l) entry is $m_{i,l} \circ N$, while MN denotes the matrix product.

The k algebra D will be generated by $\{x_1, \dots, x_j\}$ and z , where $z \in D_\alpha$ is normal. If we write D as \tilde{T}/I , then there are homogeneous relations $r_1, \dots, r_j \in I$ such that $R = (r_1, \dots, r_j)^t = F + Ez$ for a $j \times 1$ matrix E with homogeneous entries. If A is cubic then the entries in E are from $\tilde{T}_{3-\alpha}$. If A is quadratic then the entries in E are from $\tilde{T}_{2-\alpha}$. For notational convenience we will use the same letters to denote elements of the algebras T, \tilde{T}, D , and A .

Since A is AS regular of global dimension 3, there is an exact sequence S^\bullet of left A (or D) modules of the form:

2.4.

$$0 \longrightarrow A[j-6] \xrightarrow{X^t} A[j-5]^j \xrightarrow{M} A[-1]^j \xrightarrow{X} A \longrightarrow k \longrightarrow 0$$

S^3 S^2 S^1 S^0

Throughout this paper a_n will refer to the k vector space dimension of A_n and d_n will be the k vector space dimension of D_n .

Remark 2.5. Notice from 2.4 that when $j = 2$

$$a_{n+\alpha} = 2a_{n+\alpha-1} - 2a_{n+\alpha-3} + a_{n+\alpha-4} + \delta_0^{n+\alpha}$$

for all n , and when $j = 3$

$$a_{n+\alpha} = 3a_{n+\alpha-1} - 3a_{n+\alpha-2} + a_{n+\alpha-3} + \delta_0^{n+\alpha}$$

for all n .

For the convenience of the reader, an index of the notation which will be used throughout the paper is provided at the end of the paper.

3. EXTENSIONS OF CUBIC ALGEBRAS

In this section A will be cubic and $X = (x_1, x_2)^t$. Let P^\bullet be an augmented sequence of graded, projective left D modules of the form:

$$0 \rightarrow D[-\alpha-4] \xrightarrow{\Omega} D[-\alpha-3]^2 \oplus D[-4] \xrightarrow{\gamma} D[-3]^2 \oplus D[-\alpha-1]^2 \xrightarrow{\phi} D[-1]^2 \oplus D[-\alpha] \xrightarrow{\Omega^t} D \xrightarrow{\epsilon} k$$

P^4 P^3 P^2 P^1 P^0

with the usual graded augmentation map ϵ and matrices $\phi = \begin{pmatrix} M & E \\ Nz & -X \end{pmatrix}$,

$\gamma = \begin{pmatrix} -N^{-t}z & N^{-t}GM \\ X^t & E^tQ^tN^{-1} \end{pmatrix}$, and $\Omega = (x_1, x_2, z)$. In section 3.5 when z is central, the matrix N will be understood to be the identity. P_n^* will refer to the degree n part of this sequence.

Let $\pi(P^*)$ be the sequence of D modules:

$$\begin{array}{ccccccccc} \pi(P^4) & & \pi(P^3) & & \pi(P^2) & & \pi(P^1) & & \pi(P^0) \\ A[-\alpha-4] & \xrightarrow{\bar{\Omega}} & A[-\alpha-3]^2 \oplus A[-4] & \xrightarrow{\bar{\gamma}} & A[-3]^2 \oplus A[-\alpha-1]^2 & \xrightarrow{\bar{\phi}} & A[-1]^2 \oplus A[-\alpha] & \xrightarrow{\bar{\Omega}^t} & A \end{array}$$

with matrices $\bar{\Omega} = (x_1, x_2, 0)$, $\bar{\gamma} = \begin{pmatrix} 0 & N^{-t}GM \\ X^t & E^tQ^tN^{-1} \end{pmatrix}$, and $\bar{\phi} = \begin{pmatrix} M & E \\ 0 & -X \end{pmatrix}$. Notice $\bar{\phi} = \pi(\phi)$, $\bar{\Omega} = \pi(\Omega)$ and $\bar{\gamma} = \pi(\gamma)$.

Section 3.1 consists of technical lemmas, which the reader may prefer to skip.

3.1. Preliminary Lemmas.

Remark 3.1. Let $\deg(z) = \alpha$. Since $0 \rightarrow zD \hookrightarrow D \rightarrow A \rightarrow 0$ is graded exact, we see that $d_{n+\alpha} \leq d_n + a_{n+\alpha}$ with equality if and only if z is n -regular.

Lemma 3.2. *Let $\deg(z) = \alpha$. If P^* is exact in degree $n + \alpha$ and z is m -regular $\forall m < n$, then z is n -regular.*

Proof. By exactness:

$$d_{n+\alpha} = 2d_{n+\alpha-1} + d_n - 2d_{n+\alpha-3} - 2d_{n-1} + d_{n+\alpha-4} + 2d_{n-3} - d_{n-4} + \delta_0^{n+\alpha}$$

By 3.1 applied to $d_{n+\alpha-1}$, $d_{n+\alpha-3}$ and $d_{n+\alpha-4}$ and by 2.5 we obtain

$$\begin{aligned} d_{n+\alpha} &= d_n + 2(d_{n-1} + a_{n+\alpha-1}) - 2d_{n-1} - 2(d_{n-3} + a_{n+\alpha-3}) + 2d_{n-3} + (d_{n-4} + a_{n+\alpha-4}) - d_{n-4} + \delta_0^{n+\alpha} \\ &= d_n + 2a_{n+\alpha-1} - 2a_{n+\alpha-3} + a_{n+\alpha-4} + \delta_0^{n+2} = d_n + a_{n+\alpha} \end{aligned}$$

as required. \square

The following proposition ensures us that if D is an AS regular algebra then we can write $D = \tilde{T}/\langle r_1, r_2 \rangle$ where $(r_1, r_2)^t = F + Ez = R$.

Proposition 3.3. *Let $\deg(x_1) = \deg(x_2) = 1$ and $\deg(z) = \alpha$. Suppose $D = T/I$ is AS regular of global dimension 4 with $I = \langle r_1, \dots, r_{m-2}, r_{m-1} = n_1, r_m = n_2 \rangle$ for some $m \in \mathbb{N}$, such that $r_1 \equiv f_1 \pmod{z}$ and $r_2 \equiv f_2 \pmod{z}$ where f_1 and f_2 are the two cubic relations in A . Then there exists a choice of the relations r_i so that $m=4$.*

Proof. Let $e_i = \deg(r_i)$. Let g be a positive integer so that the following is a projective resolution of the trivial left D module k :

$$0 \rightarrow D[-g] \rightarrow D[-g+1]^2 \oplus D[-g+\alpha] \rightarrow \bigoplus_{i=1}^m D[-e_i] \rightarrow D[-1]^2 \oplus D[-\alpha] \rightarrow D \twoheadrightarrow k$$

Let H_D be the Hilbert series of D . Then $H_D(t)p(t) = 1$ where $p(t) = t^g - 2t^{g-1} - t^{g-\alpha} + \sum_{i=1}^m t^{e_i} - t^\alpha - 2t + 1$. Since D is infinite dimensional, $p(1) = 0$, and so $m = 4$. \square

Once we know there are four relations of degrees 3, 3, $1 + \alpha$ and $1 + \alpha$, it follows from Gorenstein symmetry that $g = \alpha + 4$. Now we show that if P^\bullet is a resolution of k , then $\text{GKdim}(D)=4$.

Proposition 3.4. *Suppose $D = \tilde{T}/\langle r_1, r_2 \rangle$ and ${}_D k$ has a projective resolution of the form:*

$$0 \rightarrow D[-\alpha-4] \longrightarrow D[-\alpha-3]^2 \oplus D[-4] \longrightarrow D[-3]^2 \oplus D[-\alpha-1]^2 \longrightarrow D[-1]^2 \oplus D[-\alpha] \rightarrow D \rightarrow k$$

Then $\text{GKdim}(D) = 4$.

Proof. The resolution implies that D has the Hilbert series

$$H_D(t) = (t^{\alpha+4} - 2t^{\alpha+3} - t^4 + 2t^{\alpha+1} + 2t^3 - t^\alpha - 2t + 1)^{-1} = (t-1)^{-3}(t+1)^{-1}(t^\alpha-1)^{-1}$$

As in [9], we know GKdim is the order of the pole of $H_D(t)$ at 1 and so we have $\text{GKdim}(D) = 4$. \square

Lemma 3.5. *P^\bullet is exact at P^1 , P^0 and k .*

Proof. The augmentation map ϵ is surjective by definition. The kernel of ϵ is the irrelevant ideal D_+ , which is covered by Ω^t . Let $v = (r_1, r_2, n_1, n_2)^t$. Notice that $\phi\Omega^t = v$ so that the image of ϕ is contained in the kernel of Ω^t . Now let $a \in P^1$ be such that $a\Omega^t = 0$ in D . Then in T , $a\Omega^t = bv + v^t c + \sum_i (h_i v)^t j_i$ where the matrix b is 4×1 , the matrices c and j_i are 1×4 and h_i is 4×4 . We can write $c = \hat{c}\Omega^t$ and $j_i = \hat{j}_i\Omega^t$ for 4×4 matrices \hat{c} and \hat{j}_i . Since $v = \phi\Omega^t$, we have in T , $a\Omega^t = b\phi\Omega^t + v^t \hat{c}\Omega^t + \sum_i (h_i v)^t \hat{j}_i\Omega^t$ which implies $a = b\phi + v^t \hat{c} + \sum_i (h_i v)^t \hat{j}_i$. Therefore in D , $a = b\phi$ and so P^\bullet is exact at P^1 . \square

Lemma 3.6. *As an element of \tilde{T} , z is 3-regular. Consequently, if there exists a matrix G in $M_2(k)$ such that $zF = GFz$ in \tilde{T}^2 , then G is invertible.*

Proof. Let σ be the automorphism of $k\langle x_1, x_2 \rangle$ defined by N^{-1} , that is, $\sigma X = N^{-1}X$. Notice that \tilde{T} is the Öre extension $k\langle x_1, x_2, \rangle[z; \sigma]$ and hence \tilde{T} is a domain. Thus z is regular in \tilde{T} , and since the f_i are linearly independent, G is invertible. \square

The following Lemma will be needed only in the cases where $\text{deg}(z) = 2$ or 3. Let C^\bullet be this sequence of left D modules:

$$0 \longrightarrow \overset{C^3}{A[-4]} \xrightarrow{X^t} \overset{C^2}{A[-3]^2} \xrightarrow{N^{-t}GM} \overset{C^1}{A[-1]^2} \xrightarrow{X} \overset{C^0}{A} \longrightarrow k \longrightarrow 0$$

Lemma 3.7. *If z is 3-regular in D and $zF = GFz$ for G in $GL_2(k)$, then C^\bullet is exact.*

Proof. It is clear that C^\bullet is exact at k , C^0 , and since A is a domain, at C^3 . To see that C^\bullet is exact at C^1 , first let a be in the image of $(N^{-t}GM)$, so $a = bN^{-t}GM$ for some $b \in C^2$. Then $aX = bN^{-t}GMX = bN^{-t}GF = 0$ in A . Now let a be in the kernel of (X) . Then since S^\bullet is exact, $a = bM$ for some $b \in S^2 = C^2$. Now notice that $bG^{-1}N^t \in C^2$ and $a = (bG^{-1}N^t)N^{-t}GM$ is in the image of $(N^{-t}GM)$, and hence C^\bullet is exact at C^1 . By dimension of C^2 it suffices to show that C^\bullet is a complex, which is clear once we show that $X^tN^{-t}GM = 0$ in A . Notice that in D , $X^tN^{-t}GMz = zX^tMN^{-1} = z(QMX)^tN^{-1} = -z(QEz)^tN^{-1} = -z(QE)^tN^{-1}z$. Now if z is 3-regular in D , $X^tN^{-t}GM = -z(QE)^tN^{-1}$ in D , so that in A , $X^tN^{-t}GM = 0$. \square

3.2. Extensions by elements of degree greater than 3.

Throughout section 3.2 $\deg(z) = \alpha$ where $\alpha \geq 4$. Recall that $\tilde{T} = k\langle x_1, x_2, z \rangle / \langle n_1, n_2 \rangle$ where $(n_1, n_2)^t = Xz - zNX$ for $N \in GL_2(k)$.

Theorem 3.8. *Let $A = D/\langle z \rangle$ be a cubic AS regular algebra of global dimension 3 defined by relations $F = MX$. Let $\alpha \geq 4$ and $z \in D_\alpha$ be normal and define N in $GL_2(k)$ by $Xz = zNX$. Let $D = \tilde{T}/\langle f_1, f_2 \rangle$. The following are equivalent:*

- (1) D is an Artin-Schelter regular algebra (of global dimension 4).
- (2) z is a regular element of D .
- (3) z is a 3-regular element of D .
- (4) There exists a matrix G in $GL_2(k)$ such that $zF = GFz$ in \tilde{T}^2 .

Remark 3.9. We notice that under the hypotheses of Theorem 3.8, if z is central then D is the polynomial extension $A[z]$ and so is always AS regular.

Proof of Theorem 3.8.

(1) \Rightarrow (2): Since z is normal in D and $D/\langle z \rangle$ is Noetherian, D is also Noetherian. By theorem 3.9 in [3], Noetherian AS-regular algebras of global dimension four are domains, and hence z is regular in D .

(2) \Rightarrow (3): Obvious.

(3) \Rightarrow (4): Let $F' = (f'_1, f'_2)^t$ for $f'_i \in k\langle x_1, x_2 \rangle_3$ be such that $zF = F'z$ in \tilde{T} . Then in D , $F'z = 0$ so $F' = 0$ since z is 3-regular. Therefore there is a matrix $G \in M_2(k)$ such that in \tilde{T} , $F' = GF$. By Lemma 3.6 G is invertible.

(4) \Rightarrow (1): Let σ be the automorphism of $k\langle x_1, x_2 \rangle$ defined by N^{-1} , that is, $\sigma X = N^{-1}X$. Condition 4 implies that σ is also an automorphism of A . Notice that D is the Öre extension $A[z; \sigma]$ and hence D is a domain and z is regular. It follows from [4] that D is Artin-Schelter

Gorenstein. From theorem 7.3.5 in [6] we know that $\text{gldim}(D) = 4$ and hence D is AS regular. In fact, using methods similar to those in section 3.3, it can be shown that P_n^* is a projective resolution of ${}_D k$ and $\text{GKdim}(D)=4$. \square

3.3. Extensions by elements of degree 3.

Throughout section 3.3 $\text{deg}(z) = 3$. Recall that $R = F + Ez = (r_1, r_2)^t$.

Theorem 3.10. *Let $A = D/\langle z \rangle$ be a cubic AS regular algebra of global dimension 3 defined by relations $F = MX$, where $(X^t M)^t = QMX$ for Q in $GL_2(k)$. Let $z \in D_3$ be normal and define N in $GL_2(k)$ by $Xz = zNX$. Let $D = \tilde{T}/\langle r_1, r_2 \rangle$ where $(r_1, r_2)^t = F + Ez$ for E in k^2 . The following are equivalent:*

- (1) D is an Artin-Schelter regular algebra (of global dimension 4).
- (2) z is a regular element of D .
- (3) z is a 3-regular element of D .
- (4) There exists a matrix G in $GL_2(k)$ such that
 - (i) $zF = GFz$ in \tilde{T}^2 ,
 - (ii) $GE = E$ in k^2 ,
 - (iii) $N^t E = QE$ in k^2 .

Corollary 3.11. *Let $z \in D_3$ be central, $A = D/\langle z \rangle$ be a cubic AS regular of global dimension 3 defined by relations $F = MX$ where $(X^t M)^t = QMX$ for Q in $GL_2(k)$, and let $D = \tilde{T}/\langle r_1, r_2 \rangle$ where $(r_1, r_2)^t = F + Ez$ for E in k^2 . The following are equivalent:*

- (1) D is an Artin-Schelter regular algebra (of global dimension 4).
- (2) z is a regular element of D .
- (3) z is a 1-regular element of D .
- (4) $E = QE$ in k^2 .

Proof of Theorem 3.10.

(1) \Rightarrow (2) and (2) \Rightarrow (3) are as in the proof of Theorem 3.8.

(3) \Rightarrow (4): Let $F' = (f'_1, f'_2)^t$ for $f'_i \in k\langle x_1, x_2 \rangle_3$ be such that $zF' = F'z$. Then $zR = z(F + Ez) = (F' + Ez)z$. Thus in D^2 , $(F' + Ez)z = 0$ so $F' + Ez = 0$ since z is 3-regular. Therefore there is a matrix $G \in M_2(k)$ such that in \tilde{T} , $F' + Ez = GR = GF + GEz$. Now push this equality into $\tilde{T}/\langle z \rangle$ to get $F' = GF$. By Lemma 3.6 G is invertible. Notice that $\tilde{T}/\langle z \rangle \cong k\langle x_1, x_2 \rangle$ which is also a subring of \tilde{T} , hence in \tilde{T}_3^2 , $F' = GF$ and $Ez = GEz$ or $E = GE$ giving conditions 4(i) and 4(ii).

The relations for D are given by $MX + Ez$, hence in D^2 we have $MX = -Ez$ and $QMX = -QEz$. This implies $X^t MX = -X^t Ez$. But $X^t M = (QMX)^t$, and so $(QMX)^t X = -X^t Ez$

which means $(QEz)^tX = (QE)^tN^{-1}Xz = X^tEz$. Now $X^tE \in D_1$ and by remark 2.3 z is 1-regular, so z can be canceled to get $(QE)^tN^{-1}X = X^tE$. But $X^tE = E^tX$ so $(QE)^tN^{-1}X = E^tX$ in D_1 . Now notice that any equality in D_1 also holds in T where X can be canceled. Thus $(QE)^tN^{-1} = E^t$ so $E = N^{-t}QE$ and we have condition (4)(iii).

(4) \Rightarrow (3): Let the ideal I of \tilde{T} be generated by (r_1, r_2) , so that $D = \tilde{T}/I$. We know by remark 3.1 that it suffices to show $d_6 \geq a_6 + d_3$. The Hilbert series of A is $H_A(t) = 1/(1+t)(1-t)^3 = 1 + 2t + 4t^2 + 6t^3 + 9t^4 + 12t^5 + 16t^6 + \dots$, so $a_6 = 16$. Clearly $\dim_k(\tilde{T}_3) = 9$ and $\dim_k(I_3) = 2$ hence $d_3 = 7$ and we must show that $d_6 \geq 23$. Recall that $\tilde{T} \cong T/\langle n_1, n_2 \rangle$. Since $\dim(T_6) = 97$ and $\langle n_1, n_2 \rangle_6$ is spanned by $\{n_i x_j x_k, x_i n_j x_k, x_i x_j n_k\}$, where $i, j, k \in \{1, 2\}$, we have $\dim_k(\tilde{T}_6) \geq 73$, and it suffices to show that $\dim_k(I_6) \leq 50$.

To do this we construct a free graded algebra S which maps onto \tilde{T} , locate a 54 dimensional subspace $B \subset S_6$ which covers I_6 , and show that the dimension of the kernel of $B \rightarrow I_6$ is at least 4. Let S be the free graded algebra on the symbols $\{X_1, X_2, Z, R_1, R_2\}$, where $\deg(X_i) = 1$ and $\deg(Z) = \deg(R_i) = 3$. Notice that if we identify X_i with x_i we get $k\langle x_1, x_2 \rangle \hookrightarrow S$. Let F_i be the image of f_i in S under this identification. We have a well defined graded ring homomorphism

$$\rho : S \longrightarrow \tilde{T}$$

given by $\rho(X_i) = x_i$, $\rho(Z) = z$ and $\rho(R_i) = r_i$. Thus we have

$$S \xrightarrow{\rho} \tilde{T} \xrightarrow{\text{mod } z} k\langle x_1, x_2 \rangle \xrightarrow{\text{mod } f_1, f_2} A$$

Let B be the subspace of S_6 with a basis of the monomials

$$\mathcal{B} := \{R_i Z, R_i X_j X_k X_l, X_i R_2 X_j X_l, X_i X_j R_2 X_l, X_i X_j X_l R_2, X_2 R_1 X_i X_j, X_i X_2 R_1 X_j, X_i X_j X_2 R_1\}$$

where $i, j, k, l \in \{1, 2\}$. Notice that $\dim_k(B) = 54$.

By conditions (4)(i) and (4)(ii) of Theorem 3.10, in \tilde{T}^2

$$zR = zF + zEz = (GF + Ez)z = (GF + GEz)z = GRz$$

and thus in the vector space \tilde{T}_6 , zr_1 and zr_2 are in the k -span of $\{r_1 z, r_2 z\}$. By condition (4)(iii), in \tilde{T} , $X^t R = X^t F + X^t E z = (QF)^t X + E^t X z = [(QF)^t + E^t N z] X = [QF + N^t E z]^t X = [QF + QEz]^t X = R^t Q^t X$, so

$$x_1 r_1 = R^t Q^t X - x_2 r_2$$

Therefore $x_1 r_1 \in k\text{-span}\{x_2 r_2, r_i x_i, r_2 x_2\}$, i.e. a spanning set for I_6 need not contain elements with $x_1 r_1$, since these can be rewritten (without using elements containing $x_1 r_2$ or $x_2 r_1$). It follows that $\rho : B \rightarrow I_6$ and hence $\dim_k(I_6) \leq 54$.

Let $\Gamma : S_6 \rightarrow S_6$ be a linear map defined on the monomial basis as follows: Γ fixes all words except those containing $X_1 R_1$. A word in S_6 which contains $X_1 R_1$ can be written as $a X_1 R_1 b$

where a and b are words in S such that $\deg(a) + \deg(b) = 2$. Since $\deg(X_1R_1) = 4$, no word in S_6 can contain more than one occurrence of X_1R_1 . Let

$$\Gamma(aX_1R_1b) = a[(R_1, R_2)Q^t(X_1, X_2)^t - X_2R_2]b$$

From this we see that the image of Γ contains no words of the form aX_1R_1 , and consequently the image of Γ^3 has no words containing X_1R_1 . Since $F_i \in S_3$, it follows that $\Gamma^3(F_iR_1)$ has no monomials containing X_1R_1 . Observe that this implies $\Gamma^3(F_iR_1) \in B$. Moreover, $\Gamma^3(F_iR_1)$ contains no words of the form $X_iX_jX_1R_2$. Also notice that as vector space maps from S_6 to \tilde{T}_6 , $\rho \circ \Gamma = \rho$, since $x_1r_1 = R^tQ^tX - x_2r_2$ in \tilde{T} .

We now exhibit 4 linearly independent elements in the kernel of $\rho : B \rightarrow I_6$ to show that $\dim_k(I_6) \leq 50$. In I_6 , we calculate that $r_1f_2 - f_1r_2 = e_1zf_2 - f_1e_2z = (e_1G_{21}f_1 + e_1G_{22}f_2 - e_2f_1)z = [(e_1G_{21} - e_2)f_1 + e_1G_{22}f_2]z$. Also $(e_1G_{21} - e_2)r_1z + e_1G_{22}r_2z = [(e_1G_{21} - e_2)f_1 + e_1G_{22}f_2]z + (e_1G_{21} - e_2)e_1z^2 + e_1G_{22}e_2z^2 = r_1f_2 - f_1r_2 + [(e_1G_{21} - e_2)e_1 + e_1G_{22}e_2]z^2$, but condition (4)(ii) implies $(e_1G_{21} - e_2)e_1 + e_1G_{22}e_2 = 0$, so that we get

$$L_1 := R_1F_2 - F_1R_2 - [(e_1G_{21} - e_2)R_1Z + e_1G_{22}R_2Z]$$

in the kernel of $B \rightarrow I_6$. Similarly we have

$$L_2 := R_2F_2 - F_2R_2 - [e_2G_{21}R_1Z + (e_2G_{22} - e_2)R_2Z]$$

in the kernel of $B \rightarrow I_6$. Also we have

$$R_1F_1 - F_1R_1 - [(e_1G_{11} - e_1)R_1Z + e_1G_{12}R_2Z]$$

$$R_2F_1 - F_2R_1 - [e_2G_{11}R_1Z + (e_2G_{12} - e_1)R_2Z]$$

in the kernel of $\rho : S_6 \rightarrow I_6$, which means

$$L_3 := R_1F_1 - \Gamma^3(F_1R_1) - [(e_1G_{11} - e_1)R_1Z + e_1G_{12}R_2Z]$$

and

$$L_4 := R_2F_1 - \Gamma^3(F_2R_1) - [e_2G_{11}R_1Z + (e_2G_{12} - e_1)R_2Z]$$

are in the kernel of $B \rightarrow I_6$, since $\rho \circ \Gamma = \rho$.

To see that the elements L_1, L_2, L_3 and L_4 are linearly independent, suppose that in S_6 , $\sum_{i=1}^4 \lambda_i L_i = 0$ for $\lambda_i \in k$. When the four relations are expressed as sums of basis elements from \mathcal{B} , only L_1 and L_2 contain monomials of the form $X_iX_jX_1R_2$, coming from F_iR_2 . Since A is a domain, no non-zero linear combination of the F_i can be factored, and thus all non-zero linear combinations of the F_iR_2 must contain monomials of the form $X_iX_jX_1R_2$ and hence $\lambda_1 = \lambda_2 = 0$.

Since no non-zero linear combination of the F_i can be factored, $\lambda_3L_3 + \lambda_4L_4$ must contain a monomial of the form $X_iX_jX_2R_1$. Since this monomial is unaffected by Γ we must have $\lambda_3 = \lambda_4 = 0$. Now we have $\dim_k(I_6) \leq 50$, proving that z is 3-regular.

(4) and (3) \Rightarrow (1) and (2):

As in 3.8, we prove this in three steps. First we show that P^\bullet is a complex. Then we show that $\pi(P^\bullet)$ is exact at $\pi(P^3)$ and $\pi(P^2)$. Finally, we prove by induction that P^\bullet is exact and z is regular.

Step One: We show that P^\bullet is a complex. By Lemma 3.5, it suffices to see that $\Omega\gamma = 0$ and $\gamma\phi = 0$.

$$\Omega\gamma = (-X^tN^{-t}z + zX^t, X^tN^{-t}GM + zE^tQ^tN^{-1})$$

By definition of N , $-X^tN^{-t}z + zX^t = 0$ in D^2 . Notice that $(X^tN^{-t}GM + zE^tQ^tN^{-1})z = z(X^tMN^{-1} + E^tQ^tN^{-1}z) = z((QMX)^tN^{-1} + (QE)^tzN^{-1}) = z(QMX + QEz)^tN^{-1} = z(QR)^tN^{-1} = 0$ in D^2 . Since z is 3-regular in D this means $X^tN^{-t}GM + zE^tQ^tN^{-1} = 0$, and so $\Omega\gamma = 0$. Now consider

$$\gamma\phi = \begin{pmatrix} -N^{-t}zM + N^{-t}GMNz & -N^{-t}zE - N^{-t}GMX \\ X^tM + E^tQ^tz & X^tE - E^tQ^tN^{-1}X \end{pmatrix}$$

By condition (4) in \tilde{T}^2 we have $zMX = (M \circ N^{-1})N^{-1}Xz = GMXz$. By Lemma ?? z is 3-regular in \tilde{T} , so $(M \circ N^{-1})N^{-1}X = GMX$ in \tilde{T}^2 and also in $k\langle x_1, x_2 \rangle^2$ where the X can be canceled to get $M \circ N^{-1} = GMN$ or $zM = GMNz$ in $M_2(\tilde{T})$, so $-N^{-t}zM + N^{-t}GMNz = N^{-t}(-zM + zM) = 0$.

By condition (4)(ii), $-N^{-t}zE - N^{-t}GMX = -N^{-t}(GMX + GEz) = -N^{-t}GR = 0$. By definition of Q , $X^tM + E^tQ^tz = (QMX)^t + (QEz)^t = R^tQ^t = 0$. By condition (4)(iii), $X^tE - E^tQ^tN^{-1}X = X^tE - X^t\tilde{E}X = 0$. Therefore $\gamma\phi = 0$ in D .

Step Two:

We show that $\pi(P^\bullet)$ is exact at $\pi(P^3)$ and $\pi(P^2)$. Since P^\bullet is a complex, $\pi(P^\bullet)$ is also a complex. As in the proof of Theorem 3.8, $\pi(P^\bullet)$ is exact at $\pi(P^3)$. To see that $\pi(P^\bullet)$ is exact at $\pi(P^2)$, let $(a, b, c, d) \in \pi(P^2)$ such that

$$(a, b, c, d)\bar{\phi} = ((a, b)M, (a, b)E - cx_1 - dx_2) = (0, 0, 0)$$

Since S^\bullet is exact at S^2 , $(a, b) = gX^t$ for some $g \in A$. So $0 = gX^tE - (c, d)X = (gE^t - (c, d))X$. By Lemma 3.7 C^\bullet is exact at C^1 so $gE^t - (c, d) = (h, l)N^{-t}GM$ for some $(h, l) \in A^2$, or $(c, d) = -(h, l)N^{-t}GM + gE^t$. Now $E^t = E^tQ^tN^{-1}$, hence $(c, d) = -(h, l)N^{-t}GM + gE^tQ^tN^{-1}$ and so $(a, b, c, d) = (-h, -l, g)\bar{\gamma}$. Thus $\pi(P^\bullet)$ is exact at $\pi(P^2)$.

Step Three: We now show by induction that P^\bullet is exact and z is regular. For the purposes of Theorem 3.10, $\alpha = 3$, however we proceed without specifying α in order to clarify how this step is applicable in the subsequent sections. By Lemma 3.5 P_n^\bullet is exact at P_n^0 and $P_n^1 \forall n$, so P_0^\bullet, P_1^\bullet and P_2^\bullet are exact since $P_n^i = 0$ if $i > 1$ and $n < 3$. Also, z is clearly 0-regular. Assume inductively that P_n^\bullet is exact and that z is m -regular $\forall m < n$. We will show that P_{n+3}^\bullet is exact and z is n -regular.

We have this canonical commutative diagram:

$$\begin{array}{ccccccc}
P^4 & & P^3 & & P^2 & & P^1 \\
D[-\alpha - 4] & \xrightarrow{\Omega} & D[-\alpha - 3]^2 \oplus D[-4] & \xrightarrow{\gamma} & D[-3]^2 \oplus D[-\alpha - 1]^2 & \xrightarrow{\phi} & D[-1]^2 \oplus D[-\alpha] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\pi(P^4) & \xrightarrow{\bar{\Omega}} & \pi(P^3) & \xrightarrow{\bar{\gamma}} & \pi(P^2) & \xrightarrow{\bar{\phi}} & \pi(P^1)
\end{array}$$

- (i) The complex $P_{n+\alpha}^\bullet$ is exact at $P_{n+\alpha}^2$: Let u be in $P_{n+\alpha}^2$ such that $u\phi = 0$. Then $\pi(u)\bar{\phi} = 0$ and $\pi(P^\bullet)$ is exact at $\pi(P^2)$ by step two above, so there exists $\bar{v} \in \pi(P_{n+\alpha}^3)$ with $\bar{v}\bar{\gamma} = \pi(u)$. Let v be in $P_{n+\alpha}^3$ with $\pi(v) = \bar{v}$. Then $\pi(u - v\gamma) = 0$, so $u = v\gamma + zw$ for some $w \in P_n^2$. Now $0 = u\phi = (v\gamma + zw)\phi = z(w\phi)$, but $w\phi \in P_n^1 = D_{n-1}^2 \oplus D_{n-\alpha}$ and z is $(n-1)$ and $(n-\alpha)$ -regular, so $w\phi = 0$. By our inductive hypothesis P_n^\bullet is exact, so there exists $w' \in P_n^3$ such that $w'\gamma = w$, and so $u = (v + zw')\gamma$.
- (ii) The complex $P_{n+\alpha}^\bullet$ is exact at $P_{n+\alpha}^3$: Let u be in $P_{n+\alpha}^3$ such that $u\gamma = 0$. Then $\pi(u)\bar{\gamma} = 0$ and $\pi(P^\bullet)$ is exact at $\pi(P^3)$, so there exists $\bar{v} \in \pi(P_{n+\alpha}^4)$ with $\bar{v}\bar{\Omega} = \pi(u)$. Let v be in $P_{n+\alpha}^4$ with $\pi(v) = \bar{v}$. Then $\pi(u - v\Omega) = 0$, so $u = v\Omega + zw$ for some $w \in P_n^3$. Now $0 = u\gamma = (v\Omega + zw)\gamma = z(w\gamma)$, but $w\gamma \in P_n^2 = D_{n-3}^2 \oplus D_{n-\alpha-1}^2$ and z is $(n-3)$ and $(n-\alpha-1)$ -regular, so $w\gamma = 0$. By hypothesis P_n^\bullet is exact, so there exists $w' \in P_n^4$ such that $w'\Omega = w$, and so $u = (v + zw')\Omega$.
- (iii) The complex $P_{n+\alpha}^\bullet$ is exact at $P_{n+\alpha}^4$: Let u be in $P_{n+\alpha}^4$ such that $u\Omega = (ux_1, ux_2, uz) = (0, 0, 0)$. Since $P_{n+\alpha}^4 = D_{n-4}$ and z is $(n-4)$ -regular, we have $u = 0$.

Therefore $P_{n+\alpha}^\bullet$ is exact. Since z is m -regular $\forall m < n$, z is n -regular by Lemma 3.2, and so by induction z is regular and P^\bullet is exact. This implies that $\text{gldim}(D) = 4$ and that D is Gorenstein. By proposition 3.4 $\text{GKdim}(D) = 4$, and so D is AS regular. \square

Proof of Corollary 3.11.

Since z is central, the matrices N and G of Theorem 3.10 are both the identity, so that condition 4(i) and 4(ii) are automatic, and condition 4(iii) reduces to $E = QE$. Now notice that in the proof of Theorem 3.10, z is only required to be 1-regular to establish condition 4(iii). \square

3.4. Extensions by elements of degree 2.

Throughout section 3.4 $\deg(z) = 2$.

Theorem 1.4. *Let $A = D/\langle z \rangle$ be a cubic AS regular algebra of global dimension 3 defined by relations $F = MX$, where $(X^t M)^t = QMX$ for Q in $\text{Gl}_2(k)$. Let $z \in D_2$ be normal and define N in $\text{GL}_2(k)$ by $Xz = zNX$. Let $(n_1, n_2) = (Xz - zNX)^t$. Write $D = \tilde{T}/\langle r_1, r_2 \rangle$ where $(r_1, r_2)^t = F + \tilde{E}Xz$ for \tilde{E} in $M_2(k)$. The following are equivalent:*

- (1) D is an Artin-Schelter regular algebra (of global dimension 4).
- (2) z is a regular element of D .
- (3) z is a 3-regular element of D .
- (4) There exists a matrix G in $GL_2(k)$ such that
 - (i) $zF = GFz$ in \tilde{T}^2 ,
 - (ii) $G\tilde{E}N = \tilde{E}$ in $M_2(k)$,
 - (iii) $N^t\tilde{E}^t = Q\tilde{E}$ in $M_2(k)$.

Corollary 3.12. *Let $z \in D_2$ be central, $A = D/\langle z \rangle$ be a cubic AS regular algebra of global dimension 3 defined by relations $F = MX$ where $(X^tM)^t = QMX$ for Q in $GL_2(k)$. Let $D = \tilde{T}/\langle r_1, r_2 \rangle$ where $(r_1, r_2)^t = F + \tilde{E}Xz$. The following are equivalent:*

- (1) D is an Artin-Schelter regular algebra (of global dimension 4).
- (2) z is a regular element of D .
- (3) z is a 2-regular element of D .
- (4) $\tilde{E}^t = Q\tilde{E}$.

Proof of Theorem 1.4.

(1) \Rightarrow (2), (2) \Rightarrow (3) and (3) \Rightarrow (4) are as in the proof of Theorem 3.8.

(4) \Rightarrow (3): Let the ideal I be generated by (r_1, r_2) , so that $D = \tilde{T}/I$. We know by remark 3.1 that it suffices to show $d_5 \geq a_5 + d_3$. The Hilbert series of A is $H_A(t) = 1/(1+t)(1-t)^3 = 1 + 2t + 4t^2 + 6t^3 + 9t^4 + 12t^5 + \dots$, so $a_5 = 12$. Clearly $\dim_k(\tilde{T}_3) = 10$ and $\dim_k(I_3) = 2$ hence $d_3 = 8$ and we must show that $d_5 \geq 20$. Equivalently, since $\dim_k(\tilde{T}_5) = 42$, it suffices to show that $\dim_k(I_5) \leq 22$.

Notice that by condition (4)(iii), $X^tR = X^tF + X^t\tilde{E}Xz = (QF)^tX + (N^{-t}Q\tilde{E}X)^t zNX = [QF + Q\tilde{E}Xz]^t X = R^tQ^tX$ in \tilde{T} . Therefore $x_1r_1 \in k\text{-span}\{x_2r_2, r_ix_j | i, j = 1, 2\}$. Notice also that by condition (4)(i) and (4)(ii) $zR = zF + z\tilde{E}Xz = (GF + \tilde{E}N^{-1}Xz)z = (GF + G\tilde{E}Xz)z = GRz$ and thus zr_1 and zr_2 are in the k -span of $\{r_1z, r_2z\}$. It follows that $I_5 \subset \text{span}\{r_iz, r_ix_jx_k, x_ir_2x_j, x_ix_jr_2, x_ix_2r_1, x_2r_1x_i\}$ so that $\dim_k(I_5) \leq 22$. Hence z is 3-regular.

(4) and (3) \Rightarrow (1) and (2):

As in the proof of Theorem 3.10 P^\bullet is a complex and $\pi(P^\bullet)$ is exact at $\pi(P^3)$ and $\pi(P^2)$. We show by induction that P^\bullet is exact and z is regular. By Lemma 3.5 P_n^\bullet is exact at P_n^0 and $P_n^1 \forall n$, so P_0^\bullet, P_1^\bullet and P_2^\bullet are exact since $P_n^i = 0$ if $i > 1$ and $n < 3$. Also, z is clearly 0-regular, and so we have the base cases for our induction. Assume inductively that P_n^\bullet is exact and that z is m -regular $\forall m < n$. We will show that P_{n+2}^\bullet is exact and z is n -regular.

We have this commutative diagram:

$$\begin{array}{ccccccccc}
P^4 & & P^3 & & P^2 & & P^1 & & P^0 \\
D[-6] & \xrightarrow{\Omega} & D[-5]^2 \oplus D[-4] & \xrightarrow{\gamma} & D[-3]^4 & \xrightarrow{\phi} & D[-1]^2 \oplus D[-2] & \xrightarrow{\Omega^t} & D & \xrightarrow{\epsilon} & k \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\pi(P^4) & \xrightarrow{\bar{\Omega}} & \pi(P^3) & \xrightarrow{\bar{\gamma}} & \pi(P^2) & \xrightarrow{\bar{\phi}} & \pi(P^1) & \xrightarrow{\bar{\Omega}^t} & \pi(P^0) & \rightarrow & k
\end{array}$$

The argument follows from this diagram exactly as in the proof of Theorem 3.10 with $\alpha = 2$. We conclude that P_{n+2}^\bullet is exact. Since z is m -regular $\forall m < n$, z is n -regular by Lemma 3.2, and so by induction z is regular and P^\bullet is exact. This implies that $\text{gldim}(D) = 4$ and that D is Gorenstein. By proposition 3.4 $\text{GKdim}(D) = 4$, and so D is AS regular. \square

Proof of Corollary 3.12.

Since z is central, the matrices N and G of Theorem 1.4 are both the identity, so that condition 4(i) and 4(ii) are automatic, and condition 4(iii) reduces to $\tilde{E}^t = Q\tilde{E}$. Now notice that in the proof of Theorem 1.4, z is only required to be 2-regular to establish condition 4(iii). \square

3.5. Extensions by elements of degree 1.

Throughout section 3.5, $\text{deg}(z) = 1$.

Remark 3.13. If D is an AS regular extension by a normal element z , then z is a regular element of D . Therefore z defines an automorphism θ of D via $\theta(a)z = za \ \forall a \in D$. If $\text{deg}(z) = 1$ we can then define a twisting system $\{\tau_m = \theta^m\}$ so that z is a central element in D^τ . By [10], D^τ is also AS regular. Notice that if we twist D^τ by τ^{-1} we get D , and so D is a twist of an AS regular central extension.

Since a normal extension by a degree 1 element is necessarily a twist by an automorphism of a central extension, we deal only with the case of central extensions.

Theorem 1.2. *Let $z \in D_1$ be central and let $A = D/\langle z \rangle$ be a cubic AS regular algebra of global dimension 3 defined by relations $F = MX$, where $(X^t M)^t = QMX$ for Q in $\text{Gl}_2(k)$. Write $D = k\langle x_1, x_2, z \rangle / \langle r_1, r_2, zx_i - x_i z \rangle$ where $(r_1, r_2)^t = F + Ez$ for E in T_2^2 . The following are equivalent:*

- (1) D is an Artin-Schelter regular algebra (of global dimension 4).
- (2) z is a regular element of D .
- (3) z is a 3-regular element of D .
- (4) $E^t Q^t X = X^t E$ in D .

Proof of Theorem 1.2.

(1) \Rightarrow (2), (2) \Rightarrow (3) and (3) \Rightarrow (4) are similar to the previous theorems.

(4) \Rightarrow (1) and (2): Here $\gamma = \begin{pmatrix} -z & 0 & M \\ 0 & -z & \\ x_1 & x_2 & E^t Q^t \end{pmatrix}$.

The centrality of z together with condition 4 imply that P^\bullet is a complex. Since S^\bullet is exact, the complex $\pi(P^\bullet)$ is exact at $\pi(P^3)$ and $\pi(P^2)$. We show by induction that P^\bullet is exact and z is regular. By Lemma 3.5, P_n^\bullet is exact at P_n^0 and $P_n^1 \ \forall n$, so P_0^\bullet and P_1^\bullet are exact. Also, z is clearly 0-regular, and so we have the base cases for our induction. Assume inductively that P_n^\bullet is exact and that z is m -regular $\forall m < n$. We will show that P_{n+1}^\bullet is exact and z is n -regular. We have this commutative diagram:

$$\begin{array}{ccccccccc} P^4 & & P^3 & & P^2 & & P^1 & & P^0 \\ D[-5] & \xrightarrow{\Omega} & D[-4]^3 & \xrightarrow{\gamma} & D[-3]^2 \oplus D[-2]^2 & \xrightarrow{\phi} & D[-1]^3 & \xrightarrow{\Omega^t} & D & \xrightarrow{\epsilon} & k \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \pi(P^4) & \xrightarrow{\bar{\Omega}} & \pi(P^3) & \xrightarrow{\bar{\gamma}} & \pi(P^2) & \xrightarrow{\bar{\phi}} & \pi(P^1) & \xrightarrow{\bar{\Omega}^t} & \pi(P^0) & \twoheadrightarrow & k \end{array}$$

The argument follows from this diagram exactly as in the proof of Theorem 3.10 with $\alpha = 1$. We conclude that P_{n+1}^\bullet is exact. Since z is m -regular $\forall m < n$, z is n -regular by Lemma 3.2, and so by induction z is regular and P^\bullet is exact. This implies that $\text{gldim}(D) = 4$ and that D is Gorenstein. By proposition 3.4 $\text{GKdim}(D) = 4$, and so D is AS regular. \square

4. EXTENSIONS OF QUADRATIC ALGEBRAS

Throughout this section A is quadratic, $X = (x_1, x_2, x_3)^t$ and P^\bullet is a graded augmented sequence of left D modules of the form

$$0 \rightarrow D[-3-\alpha] \xrightarrow{\Omega} D[-2-\alpha]^3 \oplus D[-3] \xrightarrow{\gamma} D[-2]^3 \oplus D[-1-\alpha]^3 \xrightarrow{\phi} D[-1]^3 \oplus D[-\alpha] \xrightarrow{\Omega^t} D \xrightarrow{\epsilon} k$$

with the usual graded augmentation map ϵ and matrices $\phi = \begin{pmatrix} M & E \\ Nz & -X \end{pmatrix}$,

$\gamma = \begin{pmatrix} -N^{-t}z & N^{-t}GM \\ X^t & E^t Q^t N^{-1} \end{pmatrix}$, and $\Omega = (x_1, x_2, x_3, z)$. The degree n part of this sequence will be denoted by P_n^\bullet .

Let $\pi(P^\bullet)$ be the sequence of graded left D modules:

$$\begin{array}{ccccccccc} \pi(P^4) & & \pi(P^3) & & \pi(P^2) & & \pi(P^1) & & \pi(P^0) \\ A[-\alpha-3] & \xrightarrow{\bar{\Omega}} & A[-\alpha-2]^3 \oplus A[-3] & \xrightarrow{\bar{\gamma}} & A[-2]^3 \oplus A[-\alpha-1]^3 & \xrightarrow{\bar{\phi}} & A[-1]^3 \oplus A[-\alpha] & \xrightarrow{\bar{\Omega}^t} & A \end{array}$$

with matrices $\bar{\Omega} = (x_1, x_2, x_3, 0)$, $\bar{\gamma} = \begin{pmatrix} 0 & N^{-t}GM \\ X^t & E^tQ^tN^{-1} \end{pmatrix}$, and $\bar{\phi} = \begin{pmatrix} M & E \\ 0 & -X \end{pmatrix}$. Notice that $\bar{\phi} = \pi(\phi)$, $\bar{\Omega} = \pi(\Omega)$ and $\bar{\gamma} = \pi(\gamma)$.

4.1. Preliminary Lemmas.

Notice that remark 3.1 and Lemmas 3.5 and 3.6 apply to the quadratic case as well as the cubic case.

The following Lemma ensures us that if D is an AS regular algebra then we can write $D = \tilde{T}/\langle r_1, r_2, r_3 \rangle$ where $(r_1, r_2, r_3)^t = F + Ez = R$.

Proposition 4.1. *Let $\deg(x_i) = 1$ and $\deg(z) = \alpha$. Suppose $D = T/I$ is AS regular of global dimension 4 with $I = \langle r_1, \dots, r_{m-2} = n_1, r_{m-1} = n_2, r_m = n_3 \rangle$ for some $m \in \mathbb{N}$, such that $r_i \equiv f_i \pmod{z}$ where f_1, f_2 and f_3 are the three quadratic relations in A . Then there exists a choice of the relations r_i so that $m=6$.*

Proof. Let $e_i = \deg(r_i)$. Let g be a positive integer so that the following is a projective resolution of the trivial left D module k :

$$0 \longrightarrow D[-g] \longrightarrow D[-g+1]^3 \oplus D[-g+\alpha] \longrightarrow \bigoplus_{i=1}^m D[-e_i] \longrightarrow D[-1]^3 \oplus D[-\alpha] \longrightarrow D \twoheadrightarrow k$$

Let H_D be the Hilbert series of D . Then $H_D(t)p(t) = 1$ where $p(t) = t^g - 3t^{g-1} - t^{g-\alpha} + \sum_{i=1}^m t^{e_i} - t^\alpha - 3t + 1$. Since D is infinite dimensional, $p(1) = 0$ and so $m = 6$. \square

Once we know that D has three relations of degree 2 and three relations of degree $\alpha + 1$, it follows by Gorenstein symmetry that $g = \alpha + 3$. Now we show that if P^\bullet is a resolution of k , then $\text{GKdim}(D)=4$.

Proposition 4.2. *Suppose $D = \tilde{T}/\langle r_1, r_2, r_3 \rangle$ and ${}_Dk$ has a projective resolution of the form:*

$$0 \rightarrow D[-\alpha-3] \longrightarrow D[-\alpha-2]^3 \oplus D[-3] \longrightarrow D[-2]^3 \oplus D[-\alpha-1]^3 \longrightarrow D[-1]^3 \oplus D[-\alpha] \rightarrow D \twoheadrightarrow k$$

Then $\text{GKdim}(D) = 4$.

Proof. The resolution tells us that D has the Hilbert series $H_D(t) = (t-1)^{-3}(t^\alpha-1)^{-1}$ and hence $\text{GKdim}(D) = 4$. \square

Lemma 4.3. *If $P_{n+\alpha}^\bullet$ is exact and z is m -regular $\forall m < n$, then z is n -regular.*

Proof. By exactness:

$$d_{n+\alpha} = 3d_{n+\alpha-1} + d_n - 3d_{n+\alpha-2} - 3d_{n-1} + 3d_{n-2} + d_{n+\alpha-3} - d_{n-3} + \delta_0^{n+\alpha}$$

By 3.1 applied to $d_{n+\alpha-1}$, $d_{n+\alpha-2}$ and $d_{n+\alpha-3}$ and by 2.5 we get

$$d_{n+\alpha} = d_n + 3(a_{n+\alpha-1} + d_{n-1}) - 3(a_{n+\alpha-2} + d_{n-2}) - 3d_{n-1} + 3d_{n-2} + (a_{n+\alpha-3} + d_{n-3}) - d_{n-3} + \delta_0^{n+\alpha}$$

$$= d_n + 3a_{n+\alpha-1} - 3a_{n+\alpha-2} + a_{n+\alpha-3} + \delta_0^{n+\alpha} = d_n + a_{n+\alpha}$$

as required. \square

Let C^\bullet be this sequence of D modules:

$$0 \longrightarrow A[-3] \xrightarrow{X^t} A[-2]^3 \xrightarrow{N^{-t}GM} A[-1]^3 \xrightarrow{X} A \longrightarrow k \longrightarrow 0$$

Lemma 4.4. *If z is 2-regular in D and $zF = GFz$ for G in $GL_3(k)$, then C^\bullet is exact.*

The proof of Lemma 4.4 is identical to that of Lemma 3.7.

4.2. Extensions by elements of degree greater than 2.

In this section $\deg(z) = \alpha$ where $\alpha \geq 3$.

Theorem 4.5. *Let $A = D/\langle z \rangle$ be a quadratic AS regular algebra of global dimension 3 defined by relations $F = MX$. Let $\alpha \geq 3$ and $z \in D_\alpha$ be normal and define N in $GL_3(k)$ by $Xz = zNX$. Let $D = \tilde{T}/\langle f_1, f_2, f_3 \rangle$. The following are equivalent:*

- (1) D is an Artin-Schelter regular algebra (of global dimension 4).
- (2) z is a regular element of D .
- (3) z is a 2-regular element of D .
- (4) There exists a matrix G in $Gl_3(k)$ such that $zF = GFz$ in \tilde{T}^3

Remark 4.6. We notice that under the hypotheses of Theorem 4.5, if z is central then D is the polynomial extension $A[z]$ and so is always AS regular.

The proof of Theorem 4.5 is similar to the proof of Theorem 3.8.

4.3. Extensions by elements of degree 2.

Throughout section 4.3 $\deg(z) = 2$.

Theorem 4.7. *Let $A = D/\langle z \rangle$ be a quadratic AS regular algebra of global dimension 3 defined by relations $F = MX$, where $(X^t M)^t = QMX$ for Q in $GL_3(k)$. Let $z \in D_2$ be normal and define N in $GL_3(k)$ by $Xz = zNX$. Let $D = \tilde{T}/\langle r_1, r_2, r_3 \rangle$ where $(r_1, r_2, r_3)^t = F + Ez$ for E in k^3 . The following are equivalent:*

- (1) D is an Artin-Schelter regular algebra (of global dimension 4).
- (2) z is a regular element of D .
- (3) z is a 2-regular element of D .
- (4) There exists a matrix G in $Gl_3(k)$ such that
 - (i) $zF = GFz$ in \tilde{T}^3 ,

- (ii) $GE = E$ in k^3 ,
- (iii) $N^tE = QE$ in k^3 .

Theorem 1.3. *Let $z \in D_2$ be central, $A = D/\langle z \rangle$ be a quadratic AS regular algebra of global dimension 3 defined by relations $F = MX$ where $(X^tM)^t = QMX$ for Q in $Gl_3(k)$. Write $D = k\langle x_1, x_2, x_3, z \rangle / \langle r_1, r_2, r_3, x_i z - z x_i \rangle$ where $(r_1, r_2, r_3)^t = F + Ez$ for E in k^3 . The following are equivalent:*

- (1) D is an Artin-Schelter regular algebra (of global dimension 4).
- (2) z is a regular element of D .
- (3) z is a 1-regular element of D .
- (4) $QE = E$.

Proof of Theorem 4.7.

(3) \Rightarrow (4): Let $F' = (f'_1, f'_2, f'_3)^t$ for $f'_i \in k\langle x_1, x_2, x_3 \rangle_2$ be such that $zF = F'z$. Then $zR = z(F + Ez) = (F' + Ez)z$. Thus in D , $(F' + Ez)z = 0$ or $F' + Ez = 0$ since z is 2-regular. Therefore there is a matrix $G \in M_3(k)$ such that in \tilde{T} , $F' + Ez = GR = GF + GEz$. Now push this equality into $\tilde{T}/\langle z \rangle$ to get $F' = GF$. By Lemma 3.6 G is invertible. Notice that $\tilde{T}/\langle z \rangle \cong k\langle x_1, x_2, x_3 \rangle$ which is also subring of \tilde{T} , hence in \tilde{T}_3 , $F' = GF$ and $Ez = GEz$ or $E = GE$ giving conditions 4(i) and 4(ii).

The relations for D are given by $R = MX + Ez$, hence in D^3 we have $MX = -Ez$ and $QMX = -QEz$. This implies $X^tMX = -X^tEz$. But $X^tM = (QMX)^t$, and so $(QMX)^tX = -X^tEz$ which means $(QEz)^tX = (QE)^tN^{-1}Xz = X^tEz$. Now $X^tE \in D_1$ and by remark 2.3 z is 1-regular, so z can be canceled to get $(QE)^tN^{-1}X = X^tE$. But $X^tE = E^tX$ so $(QE)^tN^{-1}X = E^tX$ in D_1 . Now notice that any equality in D_1 also holds in T where X can be canceled. Thus we have $(QE)^tN^{-1} = E^t$ so $E = N^{-t}QE$ which is condition (4)(iii).

(4) \Rightarrow (3): Let the ideal I of \tilde{T} be generated by (r_1, r_2, r_3) , so that $D = \tilde{T}/I$. We know that it suffices to show $d_4 \geq a_4 + d_2$. The Hilbert series of A is $H_A(t) = (1-t)^{-3} = 1 + 3t + 6t^2 + 10t^3 + 15t^4 + \dots$, so $a_4 = 15$. Clearly $\dim_k(\tilde{T}_2) = 10$ and $\dim_k(I_2) = 3$ hence $d_2 = 7$ and we must show that $d_6 \geq 22$. Recall that $\tilde{T} \cong T/\langle n_1, n_2, n_3 \rangle$. Since $\dim(T_4) = 109$ and $\langle n_1, n_2, n_3 \rangle_4$ is spanned by $\{n_i x_j, x_i n_j\}$, where $i, j \in \{1, 2, 3\}$, we have $\dim_k(\tilde{T}_4) \geq 91$. Since $d_4 = \dim_k(\tilde{T}_4) - \dim_k(I_4)$, it suffices to show that $\dim_k(I_4) \leq 69$.

As in the proof of Theorem 3.10, this is done by constructing a free graded algebra S which maps onto \tilde{T} , locating a 78 dimensional subspace $B \subset S_4$ which covers I_4 , and showing that the dimension of the kernel of $B \twoheadrightarrow I_4$ is at least 9. We omit this lengthy analog to the argument in Theorem 3.10.

(4) and (3) \Rightarrow (1) and (2):

As in 4.5 we prove this in three steps. First we show that P^\bullet is a complex. Then we show that $\pi(P^\bullet)$ is exact at $\pi(P^3)$ and $\pi(P^2)$. Finally, we prove by induction that P^\bullet is exact and z is regular.

Step One: P^\bullet is a complex: By Lemma 3.5, it suffices to see that $\Omega\gamma = 0$ and $\gamma\phi = 0$.

$$\Omega\gamma = (-X^tN^{-t}z + zX^t, X^tN^{-t}GM + zE^tQ^tN^{-1})$$

By definition of N , $-X^tN^{-t}z + zX^t = 0$ in D^3 . Notice that in D^3 we have $(X^tN^{-t}GM + zE^tQ^tN^{-1})z = z(X^tMN^{-1} + E^tQ^tN^{-1}z) = z((QMX)^tN^{-1} + (QE)^tzN^{-1}) = z(QMX + QEz)^tN^{-1} = z(QR)^tN^{-1} = 0$. Since z is 2-regular in D this means $X^tN^{-t}GM + zE^tQ^tN^{-1} = 0$, so $\Omega\gamma = 0$. Now consider

$$\gamma\phi = \begin{pmatrix} -N^{-t}zM + N^{-t}GMNz & -N^{-t}zE - N^{-t}GMX \\ X^tM + E^tQ^tz & X^tE - E^tQ^tN^{-1}X \end{pmatrix}$$

By condition (4) we have in \tilde{T}^3 , $zMX = (M \circ N^{-1})N^{-1}Xz = GMXz$. By Lemma 3.6 z is regular in \tilde{T} , so $(M \circ N^{-1})N^{-1}X = GMX$ in \tilde{T}^3 and also in $k\langle x_1, x_2, x_3 \rangle$ where the X can be canceled to get $zM = GMNz$. Thus in $M_3(\tilde{T})$, $-N^{-t}zM + N^{-t}GMNz = N^{-t}(-zM + zM) = 0$.

By condition (4)(ii), $-N^{-t}zE - N^{-t}GMX = -N^{-t}(GMX + GEz) = -N^{-t}GR = 0$. By definition of Q , $X^tM + E^tQ^tz = (QMX)^t + (QEz)^t = R^tQ^t = 0$. By condition (4)(iii), $X^tE - E^tQ^tN^{-1}X = X^tE - X^t\tilde{E}X = 0$. Therefore $\gamma\phi = 0$ in D .

Step Two: We show that $\pi(P^\bullet)$ is exact at $\pi(P^3)$ and $\pi(P^2)$. Since P^\bullet is a complex, $\pi(P^\bullet)$ is also a complex. As in the proof of Theorem 4.5, $\pi(P^\bullet)$ is exact at $\pi(P^3)$. To see that $\pi(P^\bullet)$ is exact at $\pi(P^2)$, let $(a, b, c, d, e, f) \in \pi(P^2)$ such that

$$(a, b, c, d, e, f)\bar{\phi} = ((a, b, c)M, (a, b, c)E - (d, e, f)X) = (0, 0, 0, 0)$$

Since S^\bullet is exact at S^2 , $(a, b, c) = gX^t$ for some $g \in A$. So $0 = gX^tE - (d, e, f)X = (gE^t - (d, e, f))X$. By Lemma 4.4 C^\bullet is exact at C^1 so $gE^t - (d, e, f) = (h, l, m)N^{-t}GM$ for some $(h, l, m) \in A^3$, or $(d, e, f) = -(h, l, m)N^{-t}GM + gE^t$. Now $E^t = E^tQ^tN^{-1}$, hence $(d, e, f) = -(h, l, m)N^{-t}GM + gE^tQ^tN^{-1}$ and so $(a, b, c, d, e, f) = (-h, -l, -m, g)\bar{\gamma}$. Thus $\pi(P^\bullet)$ is exact at $\pi(P^2)$.

Step Three: We now show by induction that P^\bullet is exact and z is regular. By Lemma 3.5 P_n^\bullet is exact at P_n^0 and $P_n^1 \ \forall n$, so P_0^\bullet, P_1^\bullet and P_2^\bullet are exact since $P_n^i = 0$ if $i > 1$ and $n < 2$. Also, z is clearly 0-regular. Assume inductively that P_n^\bullet is exact and that z is m -regular $\forall m < n$. We will show that P_{n+2}^\bullet is exact and z is n -regular.

We have this commutative diagram:

$$\begin{array}{ccccccc}
P^4 & & P^3 & & P^2 & & P^1 \\
D[-5] & \xrightarrow{\Omega} & D[-4]^3 \oplus D[-3] & \xrightarrow{\gamma} & D[-2]^3 \oplus D[-3]^3 & \xrightarrow{\phi} & D[-1]^3 \oplus D[-2] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\pi(P^4) & \xrightarrow{\bar{\Omega}} & \pi(P^3) & \xrightarrow{\bar{\gamma}} & \pi(P^2) & \xrightarrow{\bar{\phi}} & \pi(P^1)
\end{array}$$

- (i) The complex P_{n+2}^\bullet is exact at P_{n+2}^2 : Let u be in P_{n+2}^2 such that $u\phi = 0$. Then $\pi(u)\bar{\phi} = 0$ and $\pi(P^\bullet)$ is exact at $\pi(P^2)$ by step two above, so there exists $\bar{v} \in \pi(P_{n+2}^3)$ with $\bar{v}\bar{\gamma} = \pi(u)$. Let v be in P_{n+2}^3 with $\pi(v) = \bar{v}$. Then $\pi(u - v\gamma) = 0$, so $u = v\gamma + zw$ for some $w \in P_n^2$. Now $0 = u\phi = (v\gamma + zw)\phi = z(w\phi)$, but $w\phi \in P_n^1 = D_{n-1}^3 \oplus D_{n-2}$ and z is $(n-1)$ and $(n-2)$ -regular, so $w\phi = 0$. By induction P_n^\bullet is exact, so there exists $w' \in P_n^3$ such that $w'\gamma = w$, and so $u = (v + zw')\gamma$.
- (ii) The complex P_{n+2}^\bullet is exact at P_{n+2}^3 : Let u be in P_{n+2}^3 such that $u\gamma = 0$. Then $\pi(u)\bar{\gamma} = 0$ and $\pi(P^\bullet)$ is exact at $\pi(P^3)$, so there exists $\bar{v} \in \pi(P_{n+2}^4)$ with $\bar{v}\bar{\Omega} = \pi(u)$. Let v be in P_{n+2}^4 with $\pi(v) = \bar{v}$. Then $\pi(u - v\Omega) = 0$, so $u = v\Omega + zw$ for some $w \in P_n^3$. Now $0 = u\gamma = (v\Omega + zw)\gamma = z(w\gamma)$, but $w\gamma \in P_n^2 = D_{n-2}^3 \oplus D_{n-3}$ and z is $(n-2)$ and $(n-3)$ -regular, so $w\gamma = 0$. By induction P_n^\bullet is exact, so there exists $w' \in P_n^4$ such that $w'\Omega = w$, and so $u = (v + zw')\Omega$.
- (iii) The complex P_{n+2}^\bullet is exact at P_{n+2}^4 : Let u be in P_{n+2}^4 such that $u\Omega = (ux_1, ux_2, ux_3, uz) = (0, 0, 0, 0)$. Since $P_{n+2}^4 = D_{n-3}$ and z is $(n-3)$ -regular, we have $u = 0$.

We conclude that P_{n+2}^\bullet is exact. Since z is m -regular for all $m < n$, z is n -regular by Lemma 4.3, and so by induction z is regular and P^\bullet is exact. This implies that $\text{gldim}(D) = 4$ and that D is Gorenstein. By proposition 4.2 $\text{GKdim}(D) = 4$, and so D is AS regular. \square

Proof of Theorem 1.3. Notice that in the proof of Theorem 4.7 condition 4(iii) follows from the 1-regularity of z . \square

4.4. Extensions by elements of degree 1.

Throughout section 4.4 $\deg(z) = 1$. Notice that remark 3.13 applies to the quadratic case as well as the cubic case, so that normal extensions which yield AS regular algebras are all twists of central extensions.

Theorem 1.1. *Let $z \in D_1$ be central and let $A = D/\langle z \rangle$ be a quadratic AS regular algebra of global dimension 3 defined by relations $F = MX$, where $X = (x_1, x_2, x_3)^t$ and $(X^t M)^t = QMX$ for Q in $\text{Gl}_3(k)$ (cf. [1]). Write $D = k\langle x_1, x_2, x_3, z \rangle / \langle r_1, r_2, r_3, zx_i - x_i z \rangle$ where $(r_1, r_2, r_3)^t = F + Ez$ for E in $k\langle x_1, x_2, x_3 \rangle_1^3$. The following are equivalent:*

- (1) D is an Artin-Schelter regular algebra (of global dimension 4).

- (2) z is a regular element of D .
- (3) z is a 2-regular element of D .
- (4) $E^t Q^t X = X^t E$ in D .

Central extensions of quadratic AS regular algebras by elements of degree 1 were classified in [5] by the following theorem:

Theorem 4.8. (Le Bruyn, Smith, and Van den Bergh [5], Theorem 3.1.3)

Let $\deg(z) = 1$, $X = (x_1, x_2, x_3)^t$, $X^* = Q^t X$, and $D = \tilde{T}/\langle r_1, r_2, r_3 \rangle$. Write the relations for D as

$$r_j = f_j + z l_j + \alpha_j z^2 \quad j = 1, 2, 3$$

Then D is AS regular (of global dimension 4) and z is regular if and only if there exist scalars λ_j that form a solution to the following system of linear equations in $A_1^{\otimes 2}$, A_1 and k .

$$\begin{aligned} \sum_j \lambda_j f_j &= \sum_j (x_j l_j - l_j x_j^*) \\ \sum_j \lambda_j l_j &= \sum_j \alpha_j (x_j - x_j^*) \\ \sum_j \lambda_j \alpha_j &= 0 \end{aligned}$$

Proof of Theorem 1.1. (1) \Rightarrow (2), (2) \Rightarrow (3) and (3) \Rightarrow (4) are as in the previous proofs. (4) \Rightarrow (1) follows from Theorem 4.8. \square

5. EXAMPLES

Example 5.1. In this example we exhibit a global dimension 4 AS regular algebra which is a normal extension, by an element z , of an AS regular algebra, but which is not a twist by an automorphism of an extension in which z is central. We begin with an algebra A generated over \mathbb{C} in degree one by x and y with two relations:

$$f_1 = xy^2 + y^2x - 2yxy$$

$$f_2 = yx^2 + x^2y - 2xyx$$

This is the enveloping algebra of the Heisenberg Algebra, a global dimension 3 AS regular algebra of type S_1 (cf. [1]).

Let $\deg(z) = 2$ and define the algebra D as follows:

$$D = \frac{\mathbb{C}\langle x, y, z \rangle}{\langle xz - zy, yz - zx, f_1 + (x+y)z, f_2 + (x+y)z \rangle}$$

Notice that $D/\langle z \rangle \cong A$. In the language of Theorem 1.4, $M = \begin{pmatrix} y^2 & xy - 2yx \\ yx - 2xy & x^2 \end{pmatrix}$, $Q = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $N = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\tilde{E} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. If we let $G = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ then the matrix

conditions in 4 of Theorem 1.4 are satisfied, and therefore D is AS regular of global dimension 4 and z is a regular normal element of D .

Suppose that σ is a graded automorphism of D such that z is central in the twisted algebra D^σ , where multiplication in D^σ is denoted by $*$ and multiplication in D is denoted by juxtaposition. Since $x * z = z * x$ in D^σ , we have $x\sigma(z) = z\sigma(x)$ in D , which means that in A_3 , $x\overline{\sigma(z)} = 0$. Now A is a domain, so $\overline{\sigma(z)} = 0$ in A_2 , or in D , $\sigma(z) = \lambda z$ for some nonzero $\lambda \in \mathbb{C}$. The action of σ on X can be given by a 2×2 scalar matrix Σ via $\sigma(X) = \Sigma X$. In the vector space on which both D and D^σ are built, we have the equality: $\lambda Xz = X\sigma(z) = X * z = z * X = z\sigma^2(X) = z\Sigma^2 X$. Since $Xz = zNX$ in D , we have $z\lambda NX = z\Sigma^2 X$, which implies that $\lambda NX = \Sigma^2 X$ in D_1 since z is regular in D . However $D_1 \cong T_1$ where the X can be canceled to obtain $\lambda N = \Sigma^2$. We now show that no such Σ and λ can define an automorphism of D , proving that σ does not exist.

Proposition 5.2. *Let $\Sigma \in M_2(\mathbb{C})$ be such that $\Sigma^2 = \lambda N$ for nonzero $\lambda \in \mathbb{C}$. Then σ defined by $\sigma(z) = \lambda z$ and $\sigma(X) = \Sigma X$ is not an automorphism of D .*

Proof. For a contradiction, suppose we have an automorphism σ of D defined by $\sigma(z) = \lambda z$ and $\sigma(X) = \Sigma(X)$. Let $F = (f_1, f_2)^t$. Recalling that in D^2 , $F + Ez = MX + \tilde{E}Xz = 0$ we get $\sigma(F) = -\sigma(Ez) = -\lambda\tilde{E}\Sigma Xz$. Modulo z , this implies $\sigma(F) = 0$ in A^2 , so that in $\mathbb{C}\langle x, y \rangle$, $\sigma(F) = LF$ for some $L \in M_2(\mathbb{C})$. Now in D^2 , $0 = \sigma(R) = \sigma(F) + \lambda\tilde{E}\Sigma Xz$ so in \tilde{T}^2 , $\sigma(F) + \lambda\tilde{E}\Sigma Xz = \tilde{L}R = \tilde{L}F + \tilde{L}\tilde{E}Xz$ for some $\tilde{L} \in M_2(\mathbb{C})$. Modulo z this implies $LF = \tilde{L}F$ in $\mathbb{C}\langle x, y \rangle^2$, and hence $L = \tilde{L}$ since f_1 , and f_2 are linearly independent. However $\mathbb{C}\langle x, y \rangle \subset \tilde{T}$, so in \tilde{T}^2 we have both $LF = \sigma(F)$ and $L\tilde{E}Xz = \lambda\tilde{E}\Sigma Xz$. Since $F = MX$, the first equality implies

$$(*) \quad LM = (M \circ \Sigma)\Sigma$$

and, since z is regular in D , the second implies

$$(**) \quad L\tilde{E} = \lambda\tilde{E}\Sigma$$

Let $\Sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ for some $a, b, c, d \in \mathbb{C}$. It follows from $(*)$ that $L = \begin{pmatrix} ad^2 - bcd & bc^2 - acd \\ b^2c - abd & a^2d - abc \end{pmatrix}$. Since $\Sigma^2 = \lambda N$, we have the following 4 conditions on a, b, c and d :

$$a^2 + bc = 0$$

$$b(a + d) = \lambda$$

$$c(a + d) = \lambda$$

$$bc + d^2 = 0$$

In addition to these, from $(**)$ we obtain

$$ad^2 - bcd + bc^2 - acd = \lambda(a + c)$$

We now show that no choice of a, b, c and d can satisfy all five equations. Recalling that $\lambda \neq 0$, the first four equations above immediately imply $a = d, b = c, a^2 + b^2 = 0$ and $2ab = \lambda$. Then $ad^2 + bc^2 = a^3 + b^3 = (-ab)(a+b) = -\lambda(a+b)/2$ and $-bcd - acd = (-ab)(a+b) = -\lambda(a+b)/2$. Putting this into the last equation yields $-\lambda(a+b) = \lambda(a+b)$. Since $\lambda \neq 0$ and $a^2 + b^2 = 0$, there are no solutions to the system of equations, contradicting our assumption. \square

This example shows that D is not a twist by an automorphism of an extension in which z is central. It remains an open question whether, by an obscure twisting system, D is in fact a twist of a central extension.

Example 5.3. In this example we exhibit two seemingly similar algebras, although in fact only one is AS regular. We begin with an algebra A generated by degree one elements x and y with two relations:

$$\begin{aligned} f_1 &= xy^2 + y^2x + x^3 \\ f_2 &= yx^2 - x^2y \end{aligned}$$

This is a cubic global dimension 3 AS regular algebra of type S'_2 (cf. [1]).

Let $\deg(z) = 3$ and define the algebras D and D' as follows:

$$\begin{aligned} D &= \frac{k\langle x, y, z \rangle}{\langle xz - zx, yz - zy, f_1 + z, f_2 \rangle} \\ D' &= \frac{k\langle x, y, z \rangle}{\langle xz - zx, yz - zy, f_1, f_2 + z \rangle} \end{aligned}$$

Notice that both D and D' are central extensions of A . In the language of Corollary 3.11, $M = \begin{pmatrix} y^2 + x^2 & xy \\ yx & -x^2 \end{pmatrix}$, $Q = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $E = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, and $E' = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. By the matrix condition in part 4 of Corollary 3.11 we see that D is AS regular and z is a regular element of D , whereas D' is not AS regular and z is not even 1-regular in D' . One can verify that $zy = 0$ in D' .

Example 5.4. We begin with a quadratic algebra A generated by degree one elements x_1, x_2 and x_3 with three relations:

$$\begin{aligned} f_1 &= x_1^2 - x_2x_3 + x_3x_2 \\ f_2 &= x_2^2 - x_3x_1 + x_1x_3 \\ f_3 &= x_3^2 - x_1x_2 + x_2x_1 \end{aligned}$$

This is a quadratic global dimension 3 AS regular algebra of type A (cf. [1]).

Let $\deg(z) = 2$ and define the algebra D as follows:

$$D = \frac{k\langle x_1, x_2, x_3, z \rangle}{\langle x_1z - zx_2, x_2z - zx_3, x_3z - zx_1, f_1 + z, f_2 + z, f_3 + z \rangle}$$

Notice that $D/\langle z \rangle \cong A$. In the language of Theorem 4.7, $M = \begin{pmatrix} x_1 & x_3 & -x_2 \\ -x_3 & x_2 & x_1 \\ x_2 & -x_1 & x_3 \end{pmatrix}$,
 $Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $N = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ and $E = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. If we let $G = N^{-1}$ then the matrix conditions in 4 of Theorem 4.7 are satisfied, and therefore D is AS regular of global dimension 4 and z is a regular normal element of D .

Index of Notation

Symbol	Meaning	Section
k	algebraically closed field of characteristic different from two	§2
A	Artin-Schelter regular algebra of global dimension 3	§2
D	normal extension of A	§2
z	normal graded element in D	§2
F	defining relations for A , written as the vector $(f_1, \dots, f_j)^t$	§2
X	generators for A , written as the vector $(x_1, \dots, x_j)^t$	§2
M	matrix with entries from $k\langle x_1, \dots, x_j \rangle$ such that $F = MX$	§2
R	relations for D such that $R = (r_1, \dots, r_j)^t = F + Ez$	§2
E	vector with entries from $k\langle x_1, \dots, x_j, z \rangle$ such that $R = F + Ez$	§2
Q	invertible matrix with entries from k such that $X^t M = (QMX)^t$	§2
N	invertible matrix with entries from k defining the normality of z	§2
G	invertible matrix with entries from k such that $zF = GFz$	§2
π	functor from left D -modules to left A -modules by $M \mapsto M/Mz$	§2
T	free algebra on z and the generators of A	§2
\tilde{T}	$T/\langle n_1, \dots, n_j \rangle$ where $(n_1, \dots, n_j)^t = Xz - zNX$	§2
α	degree of z	§2
S^\bullet	an exact sequence of left A (or D) modules	§2
a_n	k vector space dimension of A_n	§2
d_n	k vector space dimension of D_n	§2
P^\bullet	augmented sequence of graded, projective left D -modules	§3, §4
Ω	matrix defining a map from P^4 to P^3	§3, §4
γ	matrix defining a map from P^3 to P^2	§3, §4
ϕ	matrix defining a map from P^2 to P^1	§3, §4
Ω^t	matrix defining a map from P^1 to P^0	§3, §4
$\bar{\Omega}$	matrix defining a map from $\pi(P^4)$ to $\pi(P^3)$	§3, §4
$\bar{\gamma}$	matrix defining a map from $\pi(P^3)$ to $\pi(P^2)$	§3, §4
$\bar{\phi}$	matrix defining a map from $\pi(P^2)$ to $\pi(P^1)$	§3, §4
ϵ	the augmentation map from ${}_D D$ to k	§3, §4
C^\bullet	exact sequence of left D -modules	§3.1, §4.1
σ	graded automorphism of $k\langle x_1, \dots, x_j \rangle$ and A	§3.1, §3.2, §4.2
\tilde{E}	matrix with entries from k such that $E = \tilde{E}X$	§3.4

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