1 Reading

1. Read sections 7.5, 7.6, 8.1 of Dummit and Foote

2 Problems

1. DF 7.5: 2, 3
2. DF 7.6: 3, 5c
3. DF 8.1: 4, 7, 10, 11

4. Consider the Euclidean Domain $R = \mathbb{F}_3[x]$.
   (a) Find $\gcd(x^3 + x^2 + 2, x^6 + 2x^5 + 2x^4 + 2x^2 + x + 1)$ in $R$.
   (b) Find a polynomial $f(x) \in R$ with
   
   \[
   f(x) \equiv 2x^2 + 1 \pmod{x^3 + x^2 + 2x + 2} \\
   f(x) \equiv x^3 + 2x \pmod{x^6 + 2x^5 + 2x^4 + 2x^2 + x + 1}.
   \]

5. Modify the proof that $\mathbb{Z}[i]$ is a Euclidean Domain to prove that $\mathbb{Z}[\sqrt{-2}]$ is a Euclidean Domain with respect to $N(a + b\sqrt{-2}) = a^2 + 2b^2$, and that $\mathbb{Z}[\sqrt{2}]$ is a Euclidean Domain with respect to $N(a + b\sqrt{2}) = |a^2 - 2b^2|$.

6. Let $R$ be a commutative ring and let $D \subseteq R$ be a set of non-zero elements which doesn’t contain any zero-divisors and is closed under multiplication. In class we constructed the ring of fractions $D^{-1}R$ and an injection $\psi: R \to D^{-1}R$.
   (a) Let $J \subseteq D^{-1}R$ be an ideal. Prove that there exists an ideal $I \subseteq R$ with
   
   \[
   J = D^{-1}I := \left\{ \frac{i}{d} \mid i \in I, d \in D \right\}.
   \]
   (b) Show that the map $I \mapsto D^{-1}I$ gives a one-to-one correspondence between prime ideals in $D^{-1}R$ and prime ideals in $R$ which intersect trivially with $D$.

3 Challenge Problems

Challenge Problems tend to be harder than the rest of the problems (and sometimes more interesting). You do not need to turn these in, but you should get something out of thinking about these.

1. Let $R$ be a commutative ring, and let $\mathfrak{N} \subseteq R$ be the ideal of nilpotent elements of $R$. Prove that $\mathfrak{N}$ equals the intersection of all prime ideals in $R$. (Hint: if $d \in R$ is not nilpotent then one can form the ring $D^{-1}R$ for $D = \{d^k \mid k \geq 1\}$. Use properties of this ring.)

2. Dummit and Foote 8.1.6