The following problems are mostly taken from old tests of mine. The list of problems is NOT exhaustive, you may see other types on the test. In particular, make sure you rework the homework problems that you were not able to solve right away.

The solutions to these review problems are available on the Test 3 section of our Moodle site. Work all the problems carefully. If there are parts you do not understand, read your notes or the book, ask a friend for help, or come to my office hours. Only look at the solutions once you have solved the problems.

Expect four problems on the test. This is a closed book and no calculator test. However, you will be provided with a fresh Laplace formula sheet, exactly the same as the blue one that was given out in class.

**Problem 1.** Consider the following system of linear equations

\[
\begin{align*}
3x_1 + 2x_2 - x_3 &= 4 \\
x_1 - 2x_2 + 2x_3 &= 1 \\
11x_1 + 2x_2 + x_3 &= 14
\end{align*}
\]

(a) Write this system in matrix form \( Ax = b \).

(b) Find all solutions of this system using Gaussian elimination. You need to show all relevant matrices until you have the system in row echelon form (all zeroes in the lower left triangle).

**Problem 2.** Consider the following function

\[
f(t) = \begin{cases}
0 & \text{if } t < 2 \\
t - 2 & \text{if } 2 \leq t < 5 \\
0 & \text{if } t \geq 5
\end{cases}
\]

(a) Sketch the graph of \( f(t) \) for \( 0 \leq t \leq 8 \).

(b) Write \( f \) using step functions.

(c) Find the Laplace transform of \( f(t) \).

**Problem 3.** Given is the function

\[
f(t) = 2t - t^2 + \sum_{n=1}^{\infty} (4t - 8n)u_{2n}(t), \text{ for } t \geq 0.
\]

Find the Laplace transform of this function. The solution needs to be an expression in \( s \) that does not contain \( \sum \).

**Problem 4.** A mass of 5 kg in a medium in which damping is negligible hangs on a spring with spring constant of 5 N/m. The mass is released 0.2 m below the equilibrium position (positive distance) with a downwards (positive direction) velocity of 3 m/sec. At time \( t = 6 \) sec, the mass is given a sharp blow in the upwards (negative) direction resulting in an impulse of 10 Nsec.
(a) Set up this initial value problem.
(b) Solve this initial value problem.

Problem 5. Consider the following three vectors: 
\[ x^{(1)} = \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix}, \quad x^{(2)} = \begin{pmatrix} 2 \\ -7 \\ 3 \end{pmatrix}, \quad \text{and} \quad x^{(3)} = \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix}. \]

(a) Show that these three vectors are linearly dependent and find a linear relation among them.
(b) The vector \( x^{(1)} \) is an eigenvector of the matrix 
\[ A = \begin{pmatrix} 2 & 8 & 0 \\ 3 & 7 & -1 \\ 2 & -17 & 5 \end{pmatrix}. \]
Find the corresponding eigenvalue.

Problem 6. (a) Find \( f(t) \) if its Laplace transform is 
\[ F(s) = \frac{3s}{s^2 + 4s + 13}. \]
(b) Find the Laplace transform of 
\[ f(t) = \int_0^t (t - \tau)^5 \cos(3\tau) d\tau. \]
(c) Let 
\[ A = \begin{pmatrix} -2 + i & 3 & 3 - i \\ 1 + i & 5i & -2i \end{pmatrix}. \]
1. Find \( \overline{A} \).
2. Find \( A^T \).

Problem 7. The variable \( x_1 \) denotes the position of a first mass and \( x_2 \) the position of a second mass. The two masses are connected to two walls and to themselves with three springs. This leads to the initial value problem below. Turn it into a linear system of first order differential equations.
\[ \begin{align*}
x_1'' &= -3x_1 + x_2 \\
x_2'' &= 2x_1 - 3x_2 \\
x_1(0) &= 1 \\
x_1'(0) &= 0 \\
x_2(0) &= 1.5 \\
x_2'(0) &= 0 
\end{align*} \]

Problem 8. (a) Figure 1 shows the graph of the function \( g \). Write the function \( g(t) \) using unit step functions.
(b) Calculate the Laplace transform of \( g(t) \).
(c) Find \( \mathcal{L}^{-1}\left\{ \frac{s^4 + 7}{(s+1)(s-2)} \right\} \).

Problem 9. We will use the following matrices in this problem.
\[ A = \begin{pmatrix} -9 & -2 & -10 \\ 3 & 2 & 3 \\ 8 & 2 & 9 \end{pmatrix}, \quad B = \begin{pmatrix} -4 & -2 & 3 \\ 0 & -1 & 3 \\ -2 & 5 & 3 \end{pmatrix}, \quad \vec{v} = \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}, \quad \vec{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}. \]
(a) Calculate $AB$ and $A\vec{v}$.

(b) $\vec{v}$ is an eigenvector of $A$. What is the corresponding eigenvalue?

(c) Solve the system $A\vec{x} = \vec{v}$. Show all work!

(d) Calculate $A^{-1}$.

(e) Do part (c) again, using the answer from part (d).

**Problem 10.** Below, $\mathcal{L}$ denotes the Laplace transform and $\mathcal{L}^{-1}$ the Laplace inverse transform.

(a) Find $\mathcal{L}^{-1}\left(\frac{s^2 + 21s + 28}{(s - 3)(s + 2)^2}\right)$.

(b) Here $\mathcal{L}(g(t)) = G(s)$. Find $\mathcal{L}^{-1}\left(\frac{G(s)}{(s + 3)^5}\right)$. (Leave the answer as an integral.)

(c) Calculate the Laplace transform of the triangular wave $f(t)$, $t \geq 0$ whose graph is in Figure 2.

**Problem 11.** Lead enters the body in food, air, and water contaminated by automobile and industrial excretions. It accumulates in the blood, in tissues, and, especially, in the bones. Some lead is excreted through the urinary system and by hair, nails, and sweat, but enough may remain in the body to impair mental and motor capacity.

For our model we assume, that lead is absorbed by the blood system at a constant rate of 49.3 micrograms per day. Measurements show that there is a dissipation of lead into the bones at a rate proportional to the amount of lead in the blood ($k = 0.0039$) and also into the tissues at a rate proportional to the amount of lead in the blood ($k = 0.0111$). Some lead leaves the blood by the urinary system, this rate is also proportional to the amount of lead in the blood ($k = 0.0211$).
There is a small amount of lead “washed out” from the bones back into the blood. The rate for this is proportional to the mount of lead in the bones \((k = 0.000035)\).
Likewise some lead goes back from the tissue into the blood at a rate proportional to the amount of lead in the tissue \((k = 0.0124)\). In addition, some lead leaves the tissue through hair, nails, and sweat at a rate proportional to the lead in the tissue \((k = 0.0162)\).
Let us assume that a person comes from paradise and thus has initially no lead in the system anywhere.
Define the relevant variables and set-up the system of differential equations using standard matrix notation. **Do not solve the system.**

Problem 12. *sww* denotes the sawtooth wave.

\[
f(t) = 6sww(t/10) - 2 \quad 0 \leq t \leq 100
\]

(a) Sketch this function.

(b) Use step functions and \(\sum\) notation to write out \(f(t)\) in a form that can be used by a machine.

(c) Find the Laplace transform of \(f\).

Problem 13. We study an electrical circuit to which a unit voltage impulse is applied at time \(t = \pi\). This results in the following IVP that you should solve using Laplace methods. Graph the solution.

\[
y'' + 2y' + 2y = \delta(t - \pi), \quad y(0) = 0, \quad y'(0) = 0.
\]

Problem 14. Figure 3 shows three tanks with salt mixtures that should be flushed out. At the start of this process, tank A contains 3 lbs of salt, tank B contains 2 lbs of salt and tank C 5 lbs of salt.

(a) Define appropriate variables.

(b) Use the information from the figure to set up a system of first order linear differential equations and initial values that describe the amount of salt in the tanks.

(c) Write the system in matrix form (including the initial conditions).

**Do NOT solve the problem.**

Problem 15. (a) Find the Laplace transform of \(f(t) = u_3(t)t^2\).

(b) Find \(L^{-1}\left\{\frac{4s - 17}{s^2 - 4s + 13}\right\}\).

Problem 16. Find all eigenvalues of the matrix \(A\). Then find an eigenvector to the **largest** eigenvalue. **(Do NOT find the other eigenvectors!)**

\[
A = \begin{pmatrix} 2 & 0 & 6 \\ 1 & 3 & 2 \\ 3 & 0 & -1 \end{pmatrix}
\]
Problem 17. Given are the following three vectors, where $a$ is an unknown entry.

\[
\begin{align*}
\mathbf{x}(1) &= \begin{pmatrix} -2 \\ -2 \\ 5 \end{pmatrix}, \\
\mathbf{x}(2) &= \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix}, \\
\mathbf{x}(3) &= \begin{pmatrix} -4 \\ a \\ 5 \end{pmatrix}
\end{align*}
\]

(a) Find the number $a$ that makes the three given vectors linearly dependent.

(b) Using your value of $a$ from part (a), find numbers $b$ and $c$ such that $\mathbf{x}(1) = b\mathbf{x}(2) + c\mathbf{x}(3)$.

Problem 18. Solve the initial value problem $y'' + y' + \frac{17}{4}y = 6\, \delta(t - 4)$, $y(0) = 1$ and $y'(0) = -\frac{1}{2}$.

Problem 19. Let $f(t) = u_5(t)(t - 5) - \sum_{n=6}^{\infty} u_n(t)$

(a) Sketch the graph of $f(t)$ for $0 \leq t \leq 10$.

(b) Without using the periodic function formula and keeping the summation, find $F(s) = \mathcal{L}(f(t))$.

(c) Write $F(s)$ in closed form (without the summation notation).

Problem 20. Find $\mathcal{L}^{-1}\left\{ \frac{100!}{s^{101}(s - 2)^2 + 1} \right\}$. You may leave your answer as an integral.

Problem 21. Use Laplace transforms to solve the initial value problem.

\[
y' + y = f(t), \quad y(0) = 5,
\quad \text{where } f(t) = \begin{cases} 
0, & t < \pi \\
3\cos(t - \pi), & t \geq \pi
\end{cases}
\]
Problem 22. Figure 4 is the graph of a function $f$, defined by $f(t) = 4 - t^2$ for $0 \leq t < 2$ and $f(t + 2) = f(t)$ for $t \geq 0$.

(a) Write $f(t)$ using step functions and $\sum$ notation.

(b) Find the Laplace transform of $f$.

(c) A different function, $g$, has the Laplace transform $G(s) = \frac{1}{s^2} \left(1 + 2 \sum_{k=1}^{\infty} (-1)^k e^{-ks}\right)$.

Write $G(s)$ in closed form (that is find the value of the series).

Problem 23. We have two brine (salt dissolved in water) tanks. Tank 1 initially contains 30 gallon of water with 25 oz of salt dissolved in it; and tank 2 initially contains 20 gallon of water with 15 oz of salt dissolved in it. Water containing 1 oz/gal of salt flows into tank 1 at a rate of 1.5 gal/min. The mixture flows from tank 1 to tank 2 at a rate of 3 gal/min. Water containing 3 oz/gal of salt also flows into tank 2 at a rate of 1 gal/min (from the outside). The mixture drains from tank 2 at a rate of 4 gal/min, of which some flows back into tank 1 at a rate of 1.5 gal/min, while the remainder leaves the system.

(a) Sketch the two tanks and show the flow of the salt solution into them, out of them, and between them.

(b) Define appropriate variables and set up a system of differential equations for the amount of salt in each tank and state the initial conditions. (Do not solve this initial value problem!)

(c) Find the amount of salt in each tank for which the system is in equilibrium – that is, does not change with time. Use Gaussian elimination to solve the system, bring the matrices into reduced echelon form. Do this by hand, without your calculator, showing the in-between steps.