The following problems are all from old tests of mine. Each problem was worth 20 points and there were five problems on each test. The list of problems is NOT exhaustive, you may see other types on the test. In particular, make sure you rework all the homework problems that you were not able to solve right away, including the WeBWorK problems.

The solutions to these review problems are available on the review website. Work all the problems carefully. If there are parts you do not understand, read your notes or the book, ask a friend for help, or come to my office hours. Only look at the solutions once you have solved the problems.

Work all the review problems without a calculator, without a text, or any other help. This is how it will be on the test.

**Problem 1.**

(a) Define “nonsingular matrix.”

(b) Give an example of a $3 \times 3$ matrix $A = [a_{ij}]$ with $a_{ij} \neq 0$ for all $i, j$ that is singular. Give a quick justification that your example is indeed singular.

(c) What can you say about the solution of $Ax = b$ if $A$ is nonsingular?

**Problem 2.** The maps $f, g : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ are matrix transformations. The transformation $f$ denotes a counterclockwise rotation through the angle $\pi/4$ with center at the origin; the transformation $g$ is the transformation corresponding to the matrix $B$, where

$$B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

(a) Find the matrix $A$ that corresponds to the transformation $f$.

(b) Describe the transformation $g$, sketch the segment connecting the points $(5, 1)$ and $(2, 3)$ and sketch the the image of this segment under $g$.

(c) Without doing the matrix multiplication, decide what $A^2BA^2$ is. Give a brief justification of your result.

**Problem 3.** The following transformation matrices for homogenous coordinates are given.

$$A = \begin{bmatrix} 1 & 0 & 6 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \text{and } C = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

(a) Describe in words what happens to an object that is transformed with the matrix $ABC$. Make sure you clearly indicate the order in which the transformations take place. Do NOT CALCULATE $ABC$.

(b) Calculate $CB$. Describe in words what happens to an object that is transformed with this matrix $CB$. Make sure you clearly indicate the order in which the transformations take place.
(c) Are homogeneous coordinates needed for part (a) and (b)? Explain.

Problem 4. Completely solve the system below using Gauss-Jordan elimination. Bring the system into reduced row echelon form. Show each step clearly.

\[
\begin{align*}
2x + 3y - 7z &= 9 \\
x + z &= 3 \\
2y - 6z &= 2
\end{align*}
\]

Problem 5. A large apartment building is to be built using modular construction techniques. The arrangements of apartments on any particular floor is to be chosen from one of three basic floor plans. Plan A has 18 apartments on one floor, including 3 three-bedroom units, 7 two-bedroom units, and 8 one-bedroom units. Each floor of plan B includes 4 three-bedroom units, 4 two-bedroom units, and 8 one-bedroom units. Each floor of plan C includes 5 three-bedroom units, 3 two-bedroom units, and 9 one-bedroom units. How many floors of each plan will result in a building with exactly 66 three-bedroom units, 74 two-bedroom units, and 136 one-bedroom units?

Problem 6. Let \( A \) and \( B \) be symmetric \( n \times n \) matrices.

(a) Prove that \( A + B \) is also symmetric.

(b) Make up an example of two symmetric \( 2 \times 2 \) matrices \( A \) and \( B \). Is \( AB \) of your example symmetric?

(c) Prove that \( AB \) is symmetric if and only if \( AB = BA \).

Problem 7. Consider the system of linear equations

\[
\begin{align*}
5x_1 + 3x_2 - 4x_3 + x_4 &= 0 \\
-2x_1 - 2x_3 + 2x_4 &= 0 \\
x_1 - 7x_2 + 2x_3 - 5x_4 &= 0
\end{align*}
\]

(a) Write down the coefficient matrix of this system.

(b) How do you call this type of system of linear equations?

(c) Write down the augmented matrix of this system.

(d) Without solving this problem, can you tell whether this system has a non-trivial solution or not? Give reasons for your answer.

Problem 8. In each case decide whether the statement is true or false. If it is true, just say so. If it is false, give an example that shows this. Explain why your example works (present the relevant calculations).
(a) If $A$ is an upper triangular matrix, then $A^T$ is also upper triangular.

(b) For any two $m \times n$ matrices $A$ and $B$ we have $(A + B)^T = A^T + B^T$.

(c) If $A$ is a skew symmetric matrix, then $A + A^T = 0$.

(d) For any $n \times n$ matrix $A$, if $A^2 = 0$, then $A = 0$.

(e) If the reduced row echelon form of the augmented matrix $[A|b]$ has a row consisting only of zeroes, then the system $Ax = b$ has infinitely many solutions.

Problem 9. Let $A$ be a symmetric $n \times n$ matrix whose entries are only 0's and 1's.

(a) Give an example of such a matrix $A$ if $n = 3$.

(b) Prove that if $A^2 = [c_{ij}]$ then we have $0 \leq c_{ij} \leq n$ for all $i$ and for all $j$. (NOTE: You need to show this in general, not just for your matrix in part(a)!

(c) What information does the entry $c_{ii}$ on the diagonal of $A^2$ have about the $i$-th row of $A$? Prove your statement. (NOTE: You need to show this in general, not just for your matrix in part(a)!

Problem 10. Consider the matrix $A = \begin{bmatrix} 0 & 3 & -3 & 6 \\ 1 & 0 & 2 & 3 \\ 0 & 2 & -2 & 4 \end{bmatrix}$

(a) Find a matrix $B$ that is row equivalent to $A$ and is in reduced row echelon form.

(b) Find elementary matrices $E_1 \ldots E_k$ and write $B$ as a product of these elementary matrices and $A$.

Problem 11. Consider the matrix

$A = \begin{bmatrix} 1 & 0 & -7 \\ 0 & 2 & 0 \\ 0 & 2 & \frac{3}{2} \end{bmatrix}$

(a) Find $A^{-1}$

(b) Write $A^{-1}$ as a product of elementary matrices.

(c) Write $A$ as a product of elementary matrices.

Problem 12. Consider the following system of linear equations.

$\begin{align*}
  x_1 + 2x_2 + 2x_3 &= 6 \\
  x_1 + 2x_2 + x_3 &= 5 \\
  -3x_3 &= -3
\end{align*}$
(a) Write the augmented matrix of that system.

(b) Calculate the row equivalent reduced row echelon form of the matrix from (a).

(c) Use (b) to find the solution of the given linear system.

Problem 13. One serving (30 g) of General Mills® Cheerios supplies 110 calories, 4 g of protein, 20 g of carbohydrate, and 2 g of fat. One serving (32 g) of Quaker® 100% Natural Cereal supplies 130 calories, 3 g of protein, 18 g of carbohydrate, and 5 g of fat. Suppose a mixture of these two cereals is to be prepared that contains exactly 295 calories, 9 g of protein, 48 g of carbohydrate, and 8 g of fat.

(a) Set up a system of equations to determine how many grams of each cereal should be mixed. Make sure you state what the variables in your system represent. Do NOT solve the system.

(b) Without solving the system and without looking at the particular numbers, can you tell whether such a mixture can be prepared? What is the minimum and the maximum of the number of solutions that problem might have?

Problem 14. For three positive integers m, n and r, let A be an m × n matrix with all of its entries being 1 and B an n × r matrix with also all 1’s for its entries. Prove that all entries of AB are 1.

Note: a general proof that works for all matrices of the given form is required.

Problem 15. In each case decide whether the statement is true or false. If the statement is true, just say so. If the statement is false, give a counterexample or a rigorous argument why it cannot be correct.

(a) There is a matrix A such that
    \[
    \begin{bmatrix}
    1 & 1 \\
    1 & 1
    \end{bmatrix}
    \begin{bmatrix}
    1 \\
    2
    \end{bmatrix}
    =
    \begin{bmatrix}
    1 \\
    2
    \end{bmatrix}.
    \]

(b) If A is a 2 × 3 and B a 3 × 2 matrix, then (AB)^T = A^T B^T.

(c) The matrix
    \[
    A = \begin{bmatrix}
    \sqrt{3}/2 & -1/2 \\
    1/2 & \sqrt{3}/2
    \end{bmatrix}
    \]
    rotates points in the plane by an angle of 30° around the origin in clockwise direction.

(d) If A is a 2 × 2 matrix and A^2 = 0 then A = 0.

(e) If the system Ax = b has a unique solution, then A must be a square matrix.

Problem 16. (a) Define “elementary matrix.”

(b) Give three example of elementary 3 × 3 matrices, all three of a different types.

(c) What is the connection between a non-singular matrix and the elementary matrices? Write down the most complete statement connecting the two that you know.
Problem 17. Consider the following system of linear equations.

\[
\begin{align*}
4x + 6y + 8z &= -10 \\
x + 3y - z &= -7 \\
3x + 15z &= 6
\end{align*}
\]

(a) Write the augmented matrix of that system.

(b) Calculate the row equivalent reduced row echelon form of the matrix from (a).

(c) Use (b) to find the solution of the given linear system.

Problem 18. For this problem we will need homogeneous coordinates.

(a) Write down a matrix that moves a point 5 units to the left and 2 units up.

(b) Write down a matrix that turns a point (written in homogeneous coordinates) clockwise by 90 degrees around the origin.

(c) Given is the triangle in the plane with vertices (3, 2), (1, -3), and (4, -5). Move this triangle 5 units to the left, 2 units up and then turn it clockwise by 90 degrees around the origin. Find the new coordinates using your matrices from above. Graph the new triangle in the given coordinate system.

Problem 19. In each case decide whether the statement is true or false. If the statement is true, just say so. If the statement is false, give a counterexample or a rigorous argument why it cannot be correct.

(a) A system of two linear equations in three unknowns has two solutions or infinitely many solutions.

(b) There exists a skew symmetric matrix \( \mathbf{A} \) such that \( \mathbf{A}^2 \neq \mathbf{0} \) and \( \mathbf{A}^2 \) is skew symmetric.
(c) For $2 \times 2$ matrices $A, B,$ and $C$, if $AB = AC$, then $B = C$.

(d) If an augmented matrix of a linear system in reduced row echelon form is of size $4 \times 5$ and is consistent, then the system has a unique solution.

(e) Suppose $E_1, \ldots, E_5$ are five different elementary matrices, all of the same size, $A = E_1E_2$, and $B = E_3E_4E_5$. Then $A$ and $B$ are row equivalent.

**Problem 20.** If $A = [a_{ij}]$ is an $n \times n$ matrix, then the trace of $A$, $\text{Tr} A$, is defined as

$$\text{Tr} A = \sum_{i=1}^{n} a_{ii}.$$ 

(a) Let $Z = \begin{bmatrix} -2 & 4 & 5 & -4 \\ 2 & -3 & 6 & 0 \\ -3 & 5 & 6 & 2 \\ 2 & -9 & -2 & 7 \end{bmatrix}$. Calculate $\text{Tr} Z$.

(b) Let $A$ be a skew symmetric $n \times n$ matrix with at least one non-zero entry. Prove that $\text{Tr}(A^2) < 0$.

**Note:** A general proof is needed here. You will get no points if you just show that it is correct for an example.