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If you discover errors in these solutions or feel you have a better solution, please write to us at udaepp@bucknell.edu or pgorkin@bucknell.edu. We hope that you have fun with these problems.

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Solution to Problem 12.3. (a) Let $E = (1,4)$. If $x \in (1,4)$, then $x < 4$. Therefore 4 is an upper bound. To see that 4 is the least upper bound, suppose that we have an upper bound $r$ of $E$ that is less than 4. Clearly $1 < r$. Let $s = (4 + r)/2$. Since $s$ is the average of $r$ and 4, we see that $1 < s < 4$ and $r < s$. Therefore $s \in (1,4)$ and $s > r$, so $r$ is not an upper bound. Thus, 4 is the least upper bound.

(b) Since $1.01 \in (1,4)$ and $1.01 < 1.1$, we see that 1.1 is not a lower bound.

Solution to Problem 12.6. Let $S = \{1 - 1/n : n \in \mathbb{Z}^+\}$. First, we note that $1 - 1/n \leq 1$ for every $n \in \mathbb{Z}^+$ and therefore 1 is an upper bound.

We check that 1 is the least upper bound of $S$. Note that if $r$ is an upper bound of the set with $r < 1$, then $1 - r > 0$. Therefore, there exists a positive integer $m$ with $1 - r > 1/m$. So $1 - 1/m > r$. Since $1 - 1/m$ is in our set (this is very important!) and $1 - 1/m > r$, we see that $r$ cannot be an upper bound. Therefore 1 is the least upper bound.

If you don’t know why it is essential to check that $1 - 1/m$ is in the set $S$, we suggest you work Problem 12.1.

Solution to Problem 12.9. We have assumed that $S \neq \emptyset$, so let $s \in S$. Then $\inf S \leq s \leq \sup S$, as desired. If $S = \{s\}$, then $\inf S = s = \sup S$. (Since $S$ is a finite set, the infimum is the minimum and the supremum is the maximum of the set.) If $S$ is a nonempty subset of $\mathbb{R}$ with more than one element, however, it is not possible (as you should check) for $\inf S = \sup S$.

Solution to Problem 12.12. (a) Let $U$ be an upper bound of $S$. Then $x + U \in \mathbb{R}$ and $x + s \leq x + U$ for all $s \in S$. Therefore, $x + U$ is an upper bound of $x + S$ and the set is bounded above.

(b) If we take $U$ to be the least upper bound of $S$ it is, in particular, an upper bound of $S$. From our work above, we see that $x + U$ is an upper bound of $x + S$. Since $\sup(x + S)$ is the least upper bound of $x + S$, we conclude that $\sup(x + S) \leq x + U = x + \sup S$. 
(c) Again we take \( U = \sup S \). Let \( v < x + U \). We must show that \( v \) is not an upper bound for \( x + S \).

Consider \( v - x \). Then \( U \) is the least upper bound of \( S \) and \( v - x < U \), so we see that \( v - x \) cannot be an upper bound of \( S \) ! Therefore, there exists \( s \in S \) such that \( v - x < s \). Consequently, \( v < x + s \) and \( v \) is not an upper bound of \( x + S \). This implies that there is no upper bound of \( x + S \) smaller than \( x + U \), so \( \sup(x + S) \geq x + \sup S \).

Using this and part (b) of this problem, we have \( x + U = x + \sup S \) is the least upper bound of \( x + S \); that is, \( x + \sup S = \sup(x + S) \).

Solution to Problem 12.15. First we show that \( 2 \) is an upper bound. Let \( x \in (0, 2) \). Then \( 0 < x < 2 \), so \( 2 \) is clearly an upper bound. Suppose to the contrary that \( 2 \) is not the supremum. Then there exists an upper bound \( u \) with \( u < 2 \). Since \( 1 \in (0, 1) \cap \mathbb{Q} \) we have \( 1 \leq u \). By Theorem 12.12, there is a rational number \( a \) with \( u < a < 2 \). Then \( 1 \leq u < 2 \) and \( a \in (0, 2) \cap \mathbb{Q} \).

Hence we have shown that \( u \) is not an upper bound of \( (0, 2) \cap \mathbb{Q} \) and we conclude that \( 2 \) must be the supremum.

A very similar proof shows that \( 0 \) is the infimum of the set \( (0, 2) \cap \mathbb{Q} \).

Solution to Problem 12.18. By the well-ordering principle of the natural numbers, we know that every nonempty subset of the natural numbers has a minimum. Let \( E \) be a nonempty bounded subset of the natural numbers and let \( M \) be an integer bound on \( E \). Therefore, the set \( F = \{ (M - x : x \in E) \} \) is also a nonempty subset of the natural numbers. By the well-ordering principle, \( F \) has a minimum, which we denote by \( m \). Therefore, \( m \in F \) and \( M - x \geq m \) for all \( x \in E \). Note that \( m = M - x_0 \) for some \( x_0 \in E \).

We claim that \( x_0 \) is the desired maximum.

Since \( x_0 \) is in \( E \), we need only show that it is greater than or equal to every element in \( E \). So let \( y \in E \). Then \( M - y \in F \) and consequently \( M - x_0 = m \leq M - y \). Therefore, \( -x_0 \leq -y \) or \( x_0 \geq y \), as desired.

Solution to Problem 12.21. Let \( M \in \mathbb{Z}^+ \) be chosen with \( M > |a| \). (Note that we have proved, in the corollary to the Archimedean theorem, the existence of such an integer.) Consider \( a' = a + M \) and \( b' = b + M \). Then \( 0 < a' < b' \). By the theorem, we know that there exists \( r \in \mathbb{Q} \) with \( a' < r < b' \). Therefore \( a < r - M < b \) and \( r - M \) is rational.