Solution to Problem 17.3.  (a)  \( f((-1,1)) = \{ \vert x \vert : -1 < x < 1 \} = [0,1); \)
(b)  \( f(-1,1) = \{ \vert x \vert : x = -1 \ or \ x = 1 \} = \{1\}; \)
(c)  \( f^{-1}(\{1\}) = \{ x \in \mathbb{R} : \vert x \vert = 1 \} = \{-1,1\}; \)
(d)  \( f^{-1}([-1,0)) = \{ x \in \mathbb{R} : -1 \leq \vert x \vert < 0 \} = \emptyset; \)
(e)  \( f^{-1}(f([0,1])) = f^{-1}(\{ x \in \mathbb{R} : 0 \leq \vert x \vert \leq 1 \}) = [\frac{-1}{4},\frac{1}{4}]; \)

Solution to Problem 17.6.  (a)  \( \chi_{\mathbb{Z}}(\mathbb{Z}^+) = \{ \chi_{\mathbb{Z}}(x) : x \in \mathbb{Z}^+ \} = \{1\}; \)
(b)  \( \chi_{-1}^{-1}(\mathbb{Z}^+) = \{ x \in \mathbb{R} : \chi_{\mathbb{Z}}(x) \in \mathbb{Z}^+ \} = \{ x \in \mathbb{R} : \chi_{\mathbb{Z}}(x) = 1 \} = \mathbb{Z}; \)
(c)  \( \chi_{\mathbb{Z}}(\chi_{-1}(\mathbb{Z}^+)) = \chi_{\mathbb{Z}}(\mathbb{Z}) = \{ \chi_{\mathbb{Z}}(x) : x \in \mathbb{Z} \} = \{1\}; \)
(d)  \( \chi_{-1}^{-1}(\chi_{\mathbb{Z}}(\mathbb{Z}^+)) = \chi_{-1}^{-1}(\{1\}) = \{ x \in \mathbb{R} : \chi_{\mathbb{Z}}(x) = 1 \} = \mathbb{Z}. \)

Solution to Problem 17.9.  We compute the answer using the definition of the image of a set. Now,
\[
 f(2\mathbb{Z}) = \{ f(x) : x \in 2\mathbb{Z} \} \\
 = \{ f(2m) : m \in \mathbb{Z} \} \\
 = \{ f(2m) : m \in \mathbb{Z} and m \leq 0 \} \cup \{ f(2m) : m \in \mathbb{Z} and m > 0 \} \\
 = \{ -4m : m \in \mathbb{Z} and m \leq 0 \} \cup \{ 4m - 1 : m \in \mathbb{Z} and m > 0 \} \\
 = 4\mathbb{N} \cup (4\mathbb{Z}^+ - 1) \\
 = 4\mathbb{N} \cup (4\mathbb{N} + 3). 
\]
Solution to Problem 17.12. Proof. If \( z \in f(A_1 \cup A_2) \), then there is \( x \in A_1 \cup A_2 \) such that \( z = f(x) \). If \( x \in A_1 \), then \( z = f(x) \in f(A_1) \). If \( x \not\in A_1 \), then \( x \in A_2 \). In this case \( z = f(x) \in f(A_2) \). Thus in any case, \( z = f(x) \in f(A_1) \cup f(A_2) \). This shows that \( f(A_1 \cup A_2) \subseteq f(A_1) \cup f(A_2) \).

Conversely, if \( z \in f(A_1) \cup f(A_2) \), then \( z \in f(A_1) \) or \( z \in f(A_2) \). If \( z \in f(A_1) \), then there is \( x \in A_1 \subseteq A_1 \cup A_2 \) such that \( z = f(x) \). Otherwise \( z \in f(A_2) \) and again there is \( x \in A_2 \subseteq A_1 \cup A_2 \) such that \( z = f(x) \). Thus, we conclude that \( z \in f(A_1 \cup A_2) \). This shows that \( f(A_1) \cup f(A_2) \subseteq f(A_1 \cup A_2) \).

Therefore \( f(A_1 \cup A_2) = f(A_1) \cup f(A_2) \). □

Solution to Problem 17.15. Proof. If \( x \in f^{-1}(B_1 \cap B_2) \), then \( x \in X \) and \( f(x) \in B_1 \cap B_2 \). Thus \( x \in X \) and \( f(x) \in B_1 \). This shows that \( x \in f^{-1}(B_1) \). We also have \( x \in X \) and \( f(x) \in B_2 \). This shows that \( x \in f^{-1}(B_2) \). We conclude that \( x \in f^{-1}(B_1) \cap f^{-1}(B_2) \). Hence \( f^{-1}(B_1 \cap B_2) \subseteq f^{-1}(B_1) \cap f^{-1}(B_2) \).

Conversely, if \( x \in f^{-1}(B_1) \cap f^{-1}(B_2) \), then \( x \in f^{-1}(B_1) \) and \( x \in f^{-1}(B_2) \). Thus \( x \in X \), \( f(x) \in B_1 \), and \( f(x) \in B_2 \). This implies that \( f(x) \in B_1 \cap B_2 \). Hence \( x \in f^{-1}(B_1 \cap B_2) \). We have shown that \( f^{-1}(B_1) \cap f^{-1}(B_2) \subseteq f^{-1}(B_1 \cap B_2) \).

Thus \( f^{-1}(B_1) \cap f^{-1}(B_2) = f^{-1}(B_1 \cap B_2) \). □

Solution to Problem 17.18. (a) Proof. If \( z \in f(f^{-1}(B)) \), then there is \( x \in f^{-1}(B) \) such that \( z = f(x) \). Since \( x \in f^{-1}(B) \), we conclude that \( x \in X \) and \( f(x) \in B \). Hence \( z \in B \). This proves the set inclusion.

(b) We define \( f : \mathbb{R} \to \mathbb{R} \) by \( f(x) = e^x \). Then \( f(f^{-1}(\mathbb{R})) = f(\mathbb{R}) = \mathbb{R}^+ \neq \mathbb{R} \).

(c) We claim that if the function \( f : X \to Y \) is surjective, then \( f(f^{-1}(B)) = B \).

Proof. If \( b \in B \), then there is \( x \in X \) with \( f(x) = b \) because \( f \) is onto. Thus \( x \in f^{-1}(B) \). This shows that \( b = f(x) \in f(f^{-1}(B)) \). Hence \( B \subseteq f(f^{-1}(B)) \). The reverse inclusion was proven in part (a). Therefore the two sets are equal. □

(d) The example in part (b) shows that the two sets may not be equal even if the function \( f : X \to Y \) is injective.

Solution to Problem 17.21. Since partitions are only defined for nonempty sets, we may assume that \( A \neq \emptyset \). This implies that \( B \neq \emptyset \).

Let \( b \in B \). Since the function \( f \) is onto, there exists \( a \in A \) such that \( f(a) = b \). Thus \( a \in f^{-1}(\{b\}) \). This shows that \( f^{-1}(\{b\}) \neq \emptyset \) for all \( b \in B \).

If \( a \in A \), then \( f(a) = b \in B \). Thus \( a \in f^{-1}(\{b\}) \) for \( b = f(a) \). Thus \( a \in \bigcup_{b \in B} f^{-1}(\{b\}) \). The reverse inclusion is trivial, thus \( \bigcup_{b \in B} f^{-1}(\{b\}) = A \).
Let \( f^{-1}(\{b\}) \cap f^{-1}(\{c\}) \neq \emptyset \) for some \( b, c \in B \). Hence there exists \( x \in f^{-1}(\{b\}) \cap f^{-1}(\{c\}) \). Since \( x \in f^{-1}(\{b\}) \), we have \( f(x) = b \). Also, \( x \in f^{-1}(\{c\}) \) and thus \( f(x) = c \). We conclude that \( b = c \). Hence \( f^{-1}(\{b\}) = f^{-1}(\{c\}) \).

We have thus shown that \( \{ f^{-1}(\{b\}) : b \in B \} \) partitions \( A \).

**Solution to Problem 17.24.**

(a) We claim that \( \chi_{A_1} = \chi_{A_2} \) implies \( A_1 = A_2 \). If \( x \in A_1 \), then \( \chi_{A_1}(x) = 1 \). Hence \( \chi_{A_2}(x) = 1 \). Thus \( x \in A_2 \). We have shown that \( A_1 \subseteq A_2 \). The reverse inclusion can be handled using the same argument, and the claim is then proven.

(b) If \( x \in X \), then \( x \in A_1 \cap A_2 \) or \( x \notin A_1 \cap A_2 \). In the first case, \( x \in A_1 \) and \( x \in A_2 \). Hence \( \chi_{A_1}(x) = \chi_{A_2}(x) = 1 \). In this case, \( \chi_{A_1}(x) \cdot \chi_{A_2}(x) = 1 \cdot 1 = 1 = \chi_{A_1 \cap A_2}(x) \).

In the second case \( x \notin A_1 \) or \( x \notin A_2 \). Thus \( \chi_{A_1}(x) = 0 \) or \( \chi_{A_2}(x) = 0 \). This implies that \( \chi_{A_1}(x) \cdot \chi_{A_2}(x) = 0 = \chi_{A_1 \cap A_2}(x) \).

We have shown that for all \( x \in X \), we have \( \chi_{A_1}(x) \cdot \chi_{A_2}(x) = \chi_{A_1 \cap A_2}(x) \). Hence \( \chi_{A_1} \cdot \chi_{A_2} = \chi_{A_1 \cap A_2} \).

(c) We break the proof into four cases: 1) \( x \notin A_1 \cup A_2 \), 2) \( x \in A_1 \ \setminus \ A_2 \), 3) \( x \in A_2 \ \setminus \ A_1 \), or 4) \( x \in A_1 \cap A_2 \). Note that every element of \( X \) is in exactly one of the four cases. There are sets \( A_1 \) and \( A_2 \) for which some of the cases do not occur (the corresponding sets are empty).

It is now easy to check that in all four cases \( \chi_{A_1}(x) + \chi_{A_2}(x) - \chi_{A_1 \cap A_2}(x) = \chi_{A_1 \cup A_2}(x) \). This proves the formula.

(d) Using the formula from part (c) and noticing that \( (X \ \setminus \ A_1) \cup A_1 = X \), \( (X \ \setminus \ A_1) \cap A_1 = \emptyset \), \( \chi_X = 1 \), and \( \chi_{\emptyset} = 0 \), it is straightforward to check that \( \chi_X \ \setminus A_1 = 1 - \chi_{A_1} \).