Solution to Problem 19.3.  (a) We suggest \((x_n)\), where \(x_n = \frac{n^2+5}{n+4}\). This sequence is bounded by 2, as you can check. Of course, there are many other examples.

(b) Consider \((y_n)\), defined by \(y_n = 3n - 7\). This sequence has no upper bound. Since it is an increasing sequence, \(x_n \geq x_0 = -7\) for all \(n \in \mathbb{N}\). Hence -7 is a lower bound.

(c) Consider \((z_n)\), defined by \(z_n = 1 - \frac{1}{n+1}\). This sequence is strictly increasing and \(\sup(z_n) = 1\). However, \(z_n \neq 1\) for all \(n \in \mathbb{N}\).

We claim that there does not exist a strictly increasing sequence that assumes its supremum.

Suppose to the contrary that there exists a sequence \((w_n)\) that is strictly increasing and that there exists \(m \in \mathbb{N}\) such that \(x_m = \sup(w_n)\). Then \(x_{m+1} > x_m = \sup(w_n)\). This is a contradiction and the claim is proven.

Solution to Problem 19.6. We define \((x_n)\) by

\[
x_n = \sum_{k=0}^{n} \frac{1}{k!}, \text{ for } n \in \mathbb{N}.
\]

Clearly, \(x_n \in \mathbb{Q}\) for all \(n \in \mathbb{N}\) and \((x_n)\) is increasing. From calculus we recall the Taylor series of \(e^x\) and note that \(x_n \leq e^1\) for all \(n\). Thus \((x_n)\) is increasing and bounded above, so we conclude that

\[
\sup(x_n) = \sum_{k=0}^{\infty} \frac{1}{k!} = e.
\]

(A rigorous proof that the number \(e\) is irrational can be found in Project 29.5.)
Solution to Problem 19.9. Our examples motivate us to make the following claim: If sup(\(x_n\)) = \(\ell\), then inf(\(-x_n\)) = -\(\ell\).

Proof. Since \(\ell = \sup(\ell\)), we have \(\ell \geq x_n\) for all \(n\) (in the domain of the sequence). Hence \(-\ell \leq -x_n\) for all \(n\). This shows that \(-\ell\) is a lower bound of \((-x_n)\).

Let \(u\) be a lower bound of \((-x_n)\). Then \(u \leq -x_n\) for all \(n\). Hence \(-u \geq x_n\) for all \(n\). Since \(\ell\) is the supremum of \((x_n)\), we conclude that \(\ell \leq -u\). Hence \(-\ell \geq u\). This completes the proof of the claim.

Solution to Problem 19.12. (a) Since inf\((x_n)\) \(\leq x_n\) for all \(m \in \mathbb{N}\) and inf\((y_n)\) \(\leq y_k\) for all \(k \in \mathbb{N}\), we conclude that for all \(\ell \in \mathbb{N}\) we have inf\((x_n) + inf(y_n)\) \(\leq (x_\ell + y_\ell)\). This shows that inf\((x_n) + inf(y_n)\) is a lower bound of \((x_n + y_n)\). Thus inf\((x_n) + inf(y_n)\) \(\leq inf(x_n + y_n)\).

(b) We can have strict inequality. Consider \((x_n)\) defined by \(x_n = (-1)^n\) and \((y_n)\) defined by \(y_n = (-1)^{n+1}\). Then \(x_n + y_n = 0\) for all \(n \in \mathbb{N}\). Hence
\[
inf(x_n) + inf(y_n) = -1 + (-1) = -2 < inf(x_n + y_n) = 0.
\]

Solution to Problem 19.15. (a) Since \((x_n)\) is bounded above, there exists \(M \in \mathbb{R}\) such that \(x_n \leq M\) for all \(n \in \mathbb{N}\). This implies that for all \(n \in \mathbb{N}\) we have \(y_n < x_{n+1} \leq M\). Hence \((y_n)\) is also bounded above. The completeness axiom of \(\mathbb{R}\) implies that sup\((x_n)\) and sup\((y_n)\) both exist. We claim that sup\((x_n) = sup(y_n)\). From Problem 19.14 (b) we have that sup\((x_n) \leq sup(y_n)\). By our assumptions on the two sequences, we have \(y_n < x_{n+1} \leq sup(x_n)\) for all \(n \in \mathbb{N}\). Thus sup\((x_n)\) is an upper bound for \((y_n)\). By the definition of the supremum for \((y_n)\), we have sup\((y_n) \leq sup(x_n)\). This establishes the claim.

(b) We claim that inf\((x_n)\) and inf\((y_n)\) both exist and that inf\((x_n) < inf(y_n)\).

Proof. The assumption implies that \(x_n < x_{n+1}\) for all \(n \in \mathbb{N}\). Hence \((x_n)\) is a strictly increasing sequence and thus inf\((x_n) = x_0\).

Since \(x_n < y_n\) for all \(n\), we conclude that \(x_{n+1} < y_{n+1}\) for all \(n\). Hence \(y_n < x_{n+1} < y_{n+1}\) for all \(n \in \mathbb{N}\). Thus \((y_n)\) is also strictly increasing and inf\((y_n) = y_0\). Using the assumption for the special case of \(n = 0\) we get inf\((x_n) = x_0 < y_0 = inf(y_n)\).

Solution to Problem 19.18. (a) We check that \(F_0 < F_1 \leq F_2 < F_3\), since \(F_0 = 0, F_1 = 1, F_2 = 1,\) and \(F_3 = 2\). We will further show that for \(n \geq 2\), the Fibonacci sequence is strictly increasing. This will be done using the second principle of mathematical induction (Theorem 17.6). For the base step recall that \(F_2 = 1, F_3 = 2, F_4 = 3,\) and \(F_2 < F_3 < F_4\).

For the induction step, let \(n \geq 3\) and suppose that for all integers \(m\) with \(2 \leq m \leq n\) we have \(F_{m+1} > F_m\). Then, using the induction hypothesis, we get \(F_{n+2} = F_{n+1} + F_n > F_n + F_{n-1} = F_{n+1} \).

By induction, the sequence is strictly increasing for all \(n \geq 2\) and it is increasing for all \(n \in \mathbb{N}\).
(b) We have $F_6 = 8$ and, as proven in part (a), $F_n$ is a strictly increasing sequence of integers for $n \geq 6$. Thus $F_n > n$ for $n \geq 6$. (If this is not obvious, then you can prove it with induction.) That $F_n$ is unbounded follows from Corollary 12.11.

**Solution to Problem 19.21.** Experimenting with the recursive definition leads us to the following claim: For all $n \in \mathbb{N}$, the function is defined by $f(n) = 2^{F_{n+1}}$, where $F_k$ denotes the $k$-th term of the Fibonacci sequence.

**Proof.** We will use induction to establish the claim.

For $n = 0$, we have $f(0) = 2 = 2^1 = 2^{F_1}$. For $n = 1$, we have $f(1) = 2 = 2^1 = 2^{F_2}$. Thus the formula is correct for $n = 0$ and $n = 1$.

Suppose that for some integer $n \geq 1$ and for all integers $k$, with $0 \leq k \leq n$, we know that $f(k) = 2^{F_{k+1}}$. Then

$$f(n + 1) = f(n)f(n - 1) \quad \text{(by definition of } f)$$

$$= 2^{F_{n+1}}2^{F_n} \quad \text{(by induction hypothesis)}$$

$$= 2^{F_{n+1}+F_n}$$

$$= 2^{F_{n+2}} \quad \text{(by the definition of the Fibonacci sequence).}$$

The claim follows from the second principle of mathematical induction. \qed