Solution to Problem 23.3.  
(a) A line with a rational slope is uniquely determined by its slope $m$ and its $y$-intercept $b$. Hence the set of all lines with rational slopes is equivalent to 
\[ \{ (m, b) : m \in \mathbb{Q}, b \in \mathbb{R} \} = \mathbb{Q} \times \mathbb{R} . \]
Since \( \mathbb{R} \) is uncountable and \( \mathbb{R} \approx (\{0\} \times \mathbb{R}) \subseteq \mathbb{Q} \times \mathbb{R} \), Corollary 22.4 implies that the set of all lines with rational slopes is uncountable.

(b) Since \( \mathbb{Q} = (\mathbb{Q} \setminus \{0\}) \cup \{0\} \) we conclude that \( \mathbb{Q} \setminus \{0\} \) is countably infinite.

(c) Since \( \mathbb{N} \) is countably infinite and \( \{1,3\} \) is finite, \( \mathbb{N} \setminus \{1,3\} \) is countably infinite.

(d) We can define a function $f : \mathbb{R} \to \{(x,y) \in \mathbb{R} \times \mathbb{R} : x + y = 1\}$ by $f(x) = (x, 1 - x)$. One can check that this is well-defined and bijective. Hence \( \{(x,y) \in \mathbb{R} \times \mathbb{R} : x + y = 1\} \approx \mathbb{R} \) and so this set is uncountable.

(e) In the proof of Theorem 22.12 we showed that \( (0,1) \) is uncountable. Since \( (0,1) \subseteq [0,\infty) \), Corollary 22.4 implies that \( [0,\infty) \) is uncountable.

Solution to Problem 23.6. Many of the details below appear in this chapter or previous ones. We provide the basics below.

(a) \( \mathbb{Z} \) is equivalent to \( \mathbb{N} \) and \( \mathbb{N} \subset \mathbb{Z} \).

(b) If \( A \) is countably infinite, then there is a bijection $f : \mathbb{N} \to A$. Consider $A_1 = A \setminus \{f(0)\}$. Then $f|\mathbb{Z}^+$ is a bijection of \( \mathbb{Z}^+ \) onto \( A_1 \). Since \( \mathbb{Z}^+ \) and \( \mathbb{N} \) are equivalent, there is a bijection \( g \) from \( \mathbb{Z}^+ \) onto \( \mathbb{N} \). Therefore $f \circ g \circ (f|\mathbb{Z}^+)^{-1}$ is a bijection from \( A_1 \) onto \( A \). Since \( A_1 \subset A \), this completes the example.

(c) If \( A \) is uncountable, we may choose any \( a \in A \) and consider the set \( A_a = A \setminus \{a\} \). You should provide the details that \( A_a \) must be uncountable.

(d) We have \( B \subseteq A \) and \( |A| = |B| \). Therefore, Problem 22.18 part (c) implies that \( A = B \).
Solution to Problem 23.9. We know that every subset of $\mathbb{N}$ is countable and we must show that every subset of a countable set is countable. We have already seen that every subset of a finite set is finite, so we assume that our set, $A$, is infinite. Therefore, there is a bijection $f : A \to \mathbb{N}$. Let $A_1$ be a subset of $A$. If $A_1 = \emptyset$, then it is countable, so assume $A_1 \neq \emptyset$. Then $f|A_1$ maps $A_1$ onto a subset $S$ of $\mathbb{N}$. We have already seen that the fact that $f$ is one-to-one implies $f|A_1$ is one-to-one. Therefore, $f|A_1$ is a bijection between $A_1$ and a countable set $S$. Hence $A_1$ is countable.

Solution to Problem 23.12. We think of the numbers as forming an infinite square matrix. Define $f(0)$ to be the $(1,1)$ entry. Thus, $f(0) = 1$. Now we define $f$ moving from top to bottom along the diagonal (skipping the fractions we have already defined) as follows:

- define $f(1) = 2/1$;
- define $f(2) = 1/2$;
- define $f(3) = 3/1$;
- define $f(4) = 1/3$, etc.

Then $f$ will be a bijection.

Solution to Problem 23.15. For $x \in (0,1)$, denote the decimal expansion by $x = 0.x_1x_2x_3\ldots$, where $x_j \in \{0,1,\ldots,9\}$. This expansion is unique if we decide to replace every sequence that is finite and ends in 1 with the infinite sequence ending in a string of 9’s (see the paragraph before Theorem 22.12 of the text). Define the function $f : (0,1) \to \mathbb{N}^\infty$ by $f(x) = (x_1,x_2,x_3,\ldots)$. Because of the uniqueness of the decimal expansion, this is a well-defined function. It is clearly one-to-one. Thus $(0,1) \approx \text{ran}(f) \subseteq \mathbb{N}^\infty$. Since $(0,1)$ is uncountable, Corollary 22.4 implies that $\mathbb{N}^\infty$ is uncountable.

Solution to Problem 23.18. We define $f : \mathcal{P}(A \cup B) \to \mathcal{P}(A) \times \mathcal{P}(B)$ by $f(X) = (X \cap A, X \cap B)$. First we show that $f$ is well-defined. A domain and codomain are specified. For any $X \in \mathcal{P}(A \cup B)$ we have $X \cap A \in \mathcal{P}(A)$ and $X \cap B \in \mathcal{P}(B)$. Thus $f(X)$ is defined as an element in $\mathcal{P}(A) \times \mathcal{P}(B)$. Suppose now that for some $X \in \mathcal{P}(A \cup B)$ we have $f(X) = (U,V)$ and $f(X) = (W,Z)$. Then $U = X \cap A$ and $W = X \cap A$. Thus $U = W$. Similarly, $V = Z$. Thus $(U,V) = (W,Z)$. This shows that the function is well-defined.

Next we show that $f$ is injective. So suppose that for $W,Z \in \mathcal{P}(A \cup B)$ we have $f(W) = f(Z)$; that is, $W \cap A = Z \cap A$ and $W \cap B = Z \cap B$. Note that $W \subseteq A \cup B$ and $Z \subseteq A \cup B$. Thus, if $x \in W$, then $x \in A \cup B$. Thus $x \in A$ or $x \in B$. If $x \in A$, then $x \in W \cap A = Z \cap A$. Thus $x \in Z$. If $x \in B$, then $x \in W \cap B = Z \cap B$. Thus again $x \in Z$. This shows that $W \subseteq Z$. Reversing the roles of $W$ and $Z$ shows that $W \subseteq Z$. Thus $W = Z$. This shows that $f$ is injective.

Finally we will show that $f$ is surjective. Let $(U,V) \in \mathcal{P}(A) \times \mathcal{P}(B)$. Then $U \subseteq A$ and $V \subseteq B$. Set $X = U \cup V$. Then $X \subseteq A \cup B$ and thus $X \in \text{dom}(f)$. Now $f(X) = (X \cap A, X \cap B)$. We claim that $X \cap A = U$. If $x \in X \cap A$ then $x \in U \cup V$ and $x \in A$. If $x \in V \subseteq B$, then $x \notin A$ because $A \cap B = \emptyset$. This is not possible, hence $x \in U$. We have shown that $X \cap A \subseteq U$. If $x \in U \subseteq A$, then also $x \in A$. Since $U \subseteq U \cup V = X$ we also have $x \in X$. Thus $x \in X \cap A$. Hence $U \subseteq X \cap A$. This establishes the claim. In the same way we show that $X \cap B = V$. Thus $f(X) = (U,V)$ and we have shown that $f$ is surjective.

We have now established a bijection $f : \mathcal{P}(A \cup B) \to \mathcal{P}(A) \times \mathcal{P}(B)$. This proves that $\mathcal{P}(A \cup B) \approx \mathcal{P}(A) \times \mathcal{P}(B)$. 

Solution to Problem 23.21. For our definition, \( f \) is decreasing if \( x \leq y \) implies \( f(x) \geq f(y) \). Some authors use decreasing as another way of describing “strictly decreasing.” That will change this solution slightly, but the main idea is the same.

Note that the well-ordering principle implies that every decreasing function from \( \mathbb{N} \) to \( \mathbb{N} \) is eventually constant. Therefore, we may think of this set as a subset of the set of all eventually constant sequences. (Think about this before proceeding!) Let \( E \) denote the set of eventually constant sequences (where each sequence is of the form \( (x_n)_{n \geq 1} \)). Define a map \( F : E \to \mathbb{Z} \) by

\[
F((x_n)) = 2^{x_1}3^{x_2}5^{x_3} \cdots p_m^{x_m},
\]

where \( x_m \) is the first term in the sequence for which \( x_m = x_n \) for all \( n \geq m \) and \( 2, 3, \ldots, p_m \) are the prime numbers listed in increasing order. Then you should check that \( F \) defines a one-to-one map of \( E \) into the countable set \( \mathbb{Z} \). The desired conclusion follows from this.

If you choose to use the fact that a countable union of countable sets is countable, there is another way to prove this. (See Projects 29.12 and Theorem 29.13.)

Let \( A_0 \) denote the set of all constant sequences, \( A_1 \) the set of all sequences that are constant after the first term, and, in general, let \( A_n \) denote the set of all sequences that are constant after the \( n \)-th term. We will show that \( A_n \) is countable for each \( n \).

Since \( \mathbb{N} \) is countable, the map \( f : \mathbb{N} \to A_0 \) defined by \( f(n) = (n, n, n, \ldots) \) defines a bijection between \( \mathbb{N} \) and \( A_0 \). Hence \( A_0 \) is countable.

In general, for \( A_n \), we know that \( B_n = \mathbb{N} \times \cdots \mathbb{N} \) (taken \( n \) times) is countably infinite. If we let \( g \) be a bijective map from \( \mathbb{N} \) onto \( B \) we may define \( h : \mathbb{N} \times \mathbb{N} \to A_n \) by \( h(n, m) = (g(n), m, m, \ldots) \). It is not difficult to check that \( h \) is a bijection. Since the domain is the Cartesian product of two countable sets Corollary 23.10 implies that the domain is countable. So the domain (which is obviously infinite) is countably infinite. Therefore \( A_n \) is countable for each \( n \in \mathbb{N}^+ \). The set we are interested in is equivalent a subset of \( \bigcup_n A_n \) and a countable union of countable sets is countable, we have the desired result.