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Solution to Problem 8.3. We claim that $A = \{x \in \mathbb{R} : x \geq 0\}$.

Proof. If $y \in A = \bigcup_{j=0}^{\infty} [j, j+1]$, then $y \in [j, j+1]$ for some $j \in \mathbb{N}$. Hence $y \in \mathbb{R}$ and $0 \leq j \leq y \leq j + 1$. This implies that $y \in \{x \in \mathbb{R} : x \geq 0\}$. Hence $A \subseteq \{x \in \mathbb{R} : x \geq 0\}$.

Conversely, if $y \in \{x \in \mathbb{R} : x \geq 0\}$, then we let $k = \lfloor y \rfloor$ which is defined to be the greatest integer less or equal to $y$. Since $y \geq 0$ we conclude that $k \geq 0$ and the definition of the greatest integer implies that $k \leq y < k + 1$. Hence $y \in [k, k + 1]$ for some $k \in \mathbb{N}$. We conclude that $y \in \bigcup_{j=0}^{\infty} [j, j+1] = A$. Thus $\{x \in \mathbb{R} : x \geq 0\} \subseteq A$.

The two parts together show that $A = \{x \in \mathbb{R} : x \geq 0\}$.

We claim that $B = \mathbb{Z}$.

Proof. If $y \in B = \bigcap_{j \in \mathbb{Z}} (\mathbb{R} \setminus (j, j+1))$, then $y \in \mathbb{R} \setminus (j, j+1)$ for all $j \in \mathbb{Z}$. Then $y \in \mathbb{R}$ and $y \notin (j, j+1)$ for all $j \in \mathbb{Z}$. Hence $y = k$ for some $k \in \mathbb{Z}$. That is, $y \in \mathbb{Z}$. We conclude that $B \subseteq \mathbb{Z}$.

Conversely, if $y \in \mathbb{Z}$, then $y \in \mathbb{R}$ and $y \notin (k, k + 1)$ for all $k \in \mathbb{Z}$. Hence $y \in \mathbb{R} \setminus (k, k + 1)$ for all $k \in \mathbb{Z}$. Therefore, $y \in \bigcap_{j \in \mathbb{Z}} (\mathbb{R} \setminus (j, j+1)) = B$. This means that $\mathbb{Z} \subseteq B$.

The two parts together imply that $B = \mathbb{Z}$.

Solution to Problem 8.6. This statement is false, we give a counterexample.

For $n \in \mathbb{Z}^+$ we define $A_n = [0, 1/n)$ and $B_n = [0, 1/n]$, the half open and closed interval of reals, respectively. It is clear that $A_n \subset B_n$ for all $n \in \mathbb{Z}^+$. It is also easily seen that

$$\bigcap_{n \in \mathbb{Z}^+} A_n = \bigcap_{n \in \mathbb{Z}^+} B_n = \{0\}.$$
Solution to Problem 8.9. If \( x \in \bigcup_{n \in \mathbb{Z}^+} A_n \), then \( x \in A_{n_0} \) for some \( n_0 \in \mathbb{Z}^+ \); that is,
\[
0 < \frac{1}{n_0} < x \leq 2 < \frac{3}{n} + 2 \quad \text{for all} \ n \in \mathbb{Z}^+.
\]
Hence \( x \in B_n \) for all \( n \in \mathbb{Z}^+ \). Thus, \( x \in \bigcap_{n \in \mathbb{Z}^+} B_n \) and \( \bigcup_{n \in \mathbb{Z}^+} A_n \subseteq \bigcap_{n \in \mathbb{Z}^+} B_n \).

Solution to Problem 8.12. (a) If \( x \in \left( \bigcup_{\alpha \in \mathcal{I}} A_\alpha \right) \cap B \), then \( x \in \bigcup_{\alpha \in \mathcal{I}} A_\alpha \) and \( x \in B \). Thus \( x \in A_\alpha \) for some \( \alpha \in \mathcal{I} \) and \( x \in B \). This implies that \( x \in A_\alpha \cap B \) for some \( \alpha \in \mathcal{I} \). Hence \( x \in \bigcup_{\alpha \in \mathcal{I}} (A_\alpha \cap B) \).

Consequently, \( \left( \bigcup_{\alpha \in \mathcal{I}} A_\alpha \right) \cap B \subseteq \bigcup_{\alpha \in \mathcal{I}} (A_\alpha \cap B) \).

Conversely, if \( x \in \bigcup_{\alpha \in \mathcal{I}} (A_\alpha \cap B) \), then \( x \in A_\alpha \cap B \) for some \( \alpha \in \mathcal{I} \). Hence \( x \in A_\alpha \) for some \( \alpha \in \mathcal{I} \) and \( x \in B \). Thus, \( x \in \bigcup_{\alpha \in \mathcal{I}} A_\alpha \) and \( x \in B \). That is, \( x \in \left( \bigcup_{\alpha \in \mathcal{I}} A_\alpha \right) \cap B \). This shows that \( \bigcup_{\alpha \in \mathcal{I}} (A_\alpha \cap B) \subseteq \left( \bigcup_{\alpha \in \mathcal{I}} A_\alpha \right) \cap B \).

The two parts together show the equality of the two sets.

(b) We claim that under the given conditions the following set equality holds:
\[
\left( \bigcap_{\alpha \in \mathcal{I}} A_\alpha \right) \cup B = \bigcap_{\alpha \in \mathcal{I}} (A_\alpha \cup B).
\]

If \( x \in \left( \bigcap_{\alpha \in \mathcal{I}} A_\alpha \right) \cup B \), then \( x \in \bigcap_{\alpha \in \mathcal{I}} A_\alpha \) or \( x \in B \). Hence \( x \in A_\alpha \) for all \( \alpha \in \mathcal{I} \) or \( x \in B \). This implies that \( x \in A_\alpha \cup B \) for all \( \alpha \in \mathcal{I} \). (Why?) Thus, \( x \in \bigcap_{\alpha \in \mathcal{I}} (A_\alpha \cup B) \). This shows that \( \left( \bigcap_{\alpha \in \mathcal{I}} A_\alpha \right) \cup B \subseteq \bigcap_{\alpha \in \mathcal{I}} (A_\alpha \cup B) \).

Conversely, if \( x \in \bigcap_{\alpha \in \mathcal{I}} (A_\alpha \cup B) \), then \( x \in A_\alpha \cup B \) for all \( \alpha \in \mathcal{I} \). Thus \( x \in B \) or \( x \in A_\alpha \) for all \( \alpha \in \mathcal{I} \). Hence \( x \in B \) or \( x \in \bigcap_{\alpha \in \mathcal{I}} A_\alpha \). (Why?) This implies that \( x \in \left( \bigcap_{\alpha \in \mathcal{I}} A_\alpha \right) \) \cup B \). Consequently, \( \bigcap_{\alpha \in \mathcal{I}} (A_\alpha \cup B) \subseteq \left( \bigcap_{\alpha \in \mathcal{I}} A_\alpha \right) \cup B \).

The two parts together establish the claim.

Solution to Problem 8.15. Claim: \( A = 2\mathbb{Z} \).

Proof. If \( x \in 2\mathbb{Z} \), then \( x \in \mathbb{Q} \). Also, \( x = 2m \) for some \( m \in \mathbb{Z} \). Thus \( x \not\in \mathbb{R} \setminus \{2m\} \) for some \( m \in \mathbb{Z} \). Hence \( x \not\in \bigcap_{n \in \mathbb{Z}} (\mathbb{R} \setminus \{2n\}) \). Hence \( x \in \mathbb{Q} \setminus \bigcap_{n \in \mathbb{Z}} (\mathbb{R} \setminus \{2n\}) = A \). Thus \( 2\mathbb{Z} \subseteq A \).

Conversely, if \( x \in A \), then \( x \in \mathbb{Q} \setminus \bigcap_{n \in \mathbb{Z}} (\mathbb{R} \setminus \{2n\}) \). That is, \( x \in \mathbb{Q} \) and \( x \not\in \bigcap_{n \in \mathbb{Z}} (\mathbb{R} \setminus \{2n\}) \). Thus \( x \not\in \mathbb{R} \setminus \{2n\} \) for some \( n \in \mathbb{Z} \). Since \( x \in \mathbb{Q} \subseteq \mathbb{R} \), we conclude that that \( x \in \{2n\} \) for some \( n \in \mathbb{Z} \). That is, \( x = 2n \) for some \( n \in \mathbb{Z} \). Hence \( x \in 2\mathbb{Z} \). This shows that \( A \subseteq 2\mathbb{Z} \).

This finishes the proof that \( A = 2\mathbb{Z} \). \( \square \)
Solution to Problem 8.18.  
(a) Let \( A_n = \{x \in \mathbb{R} : n \leq x < n + 1\} \). Then the collection \( \mathcal{A} = \{A_n : n \in \mathbb{Z}^+\} \) is pairwise disjoint: If \( A_n, A_m \in \mathcal{A} \) with \( A_n \cap A_m \neq \emptyset \), then there is \( x \in A_n \cap A_m \). This implies that \( n \leq x < m + 1 \). Hence \( n - m < 1 \). Since both \( m \) and \( n \) are positive integers, we conclude that \( n = m \). Hence \( A_n = A_m \). This shows that \( \mathcal{A} \) is pairwise disjoint.

(b) The contrapositive is: “If \( X \neq Y \), then \( X \cap Y = \emptyset \).”

(c) The converse is: “If \( X = Y \), then \( X \cap Y \neq \emptyset \).”

(d) Yes, it does hold. The reason is that the assertion of (b) is the contrapositive of the defining condition of pairwise disjoint collection. The contrapositive is logically equivalent with the original statement.

(e) Yes the set \( \mathcal{A} \) is a pairwise disjoint collection. The statement of part (b) is equivalent to the defining statement of pairwise disjoint collection.

(f) In the trivial case, \( \mathcal{B} = \{B\} \), where \( B \neq \emptyset \), this is false. In all other cases, this is true. We begin with the trivial case. In that case, \( \mathcal{B} = \{B\} \), where \( B \neq \emptyset \), thus \( \mathcal{B} \) is pairwise disjoint. But we have \( \bigcap_{X \in \mathcal{B}} X = B \neq \emptyset \). (Recall that we have to assume that \( \mathcal{B} \neq \emptyset \) because the intersection of an empty collection of sets is not defined!)

However, if \( \mathcal{B} \) has at least two elements, then \( \bigcap_{X \in \mathcal{B}} X = \emptyset \): Suppose not, then there is \( x \in \bigcap_{X \in \mathcal{B}} X \). Let \( X_1 \) and \( X_2 \) be two elements of \( \mathcal{B} \). (We may assume that they are different because a set that has two elements, both the same, is not considered to have two elements.) Then \( x \in X_1 \) and \( x \in X_2 \). This contradicts the condition of being pairwise disjoint.

(g) No, it need not be pairwise disjoint. Consider \( \mathcal{B} = \{[0, 3], [2, 5], [4, 7]\} \). Here the sets denote closed intervals of the reals. Then \( [0, 3] \cap [2, 5] = [2, 3] \neq \emptyset \) and thus \( \mathcal{B} \) is not pairwise disjoint. But \( \bigcap_{X \in \mathcal{B}} = [0, 3] \cap [2, 5] \cap [4, 7] = \emptyset \).

Solution to Problem 8.21. There are many examples. For example, let \( A_j = [-1/j, \infty) = \{x \in \mathbb{R} : x \geq -1/j\} \). We clearly have \( A_{j+1} \subset A_j \) for all \( j \in \mathbb{Z}^+ \). Also, \( \bigcap_{j=1}^{\infty} A_j = \{x \in \mathbb{R} : x \geq 0\} \neq \emptyset \).